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Topics in quantum topology

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TOPICS IN QUANTUM TOPOLOGY

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
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Doctor of Philosophy

in

The Department of Mathematics

by

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Dedicated to:

My parents

my wife, and

my family

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I praise GOD, the most merciful, for his gracious help in finishing this work.

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Table of Contents

| | |
|---|-----------|
| Acknowledgments | iii |
| Abstract | v |
| Chapter 1 The Parity of the Maslov Index and the Even Cobordism Category | 1 |
| 1.1 Introduction | 1 |
| 1.2 Lagrangian Subspaces and the Maslov Index | 2 |
| 1.3 The Weighted Cobordism Categories | 7 |
| 1.4 The Even Cobordism Category | 10 |
| Chapter 2 Integral Bases for Certain TQFT Modules of the Torus | 13 |
| 2.1 Introduction | 13 |
| 2.2 The $SO(3)$ - and $SU(2)$ -TQFTs | 14 |
| 2.3 The Integral Cobordism Functor | 15 |
| 2.4 The First Basis for $\mathcal{S}_p(S^1 \times S^1)$ | 18 |
| 2.5 The Frohman Kania-Bartoszyńska Ideal | 21 |
| 2.6 The Quantization Functors for $p = 1$, and 2 | 22 |
| 2.7 Relating the r -th and $2r$ -th Theories | 27 |
| 2.8 The Second Basis for $\mathcal{S}_p(S^1 \times S^1)$ | 31 |
| Chapter 3 Periodic 3-Manifolds and Modular Categories | 35 |
| 3.1 Introduction | 35 |
| 3.2 Quantum Invariants of 3-Manifolds..... | 36 |
| 3.2.1 Introduction | 36 |
| 3.2.2 The τ -Invariant of Closed 3-Manifolds | 36 |
| 3.2.3 The I -Invariant | 37 |
| 3.3 Periodic Links and Periodic 3-Manifolds | 37 |
| 3.4 Some Results About Traces | 38 |
| 3.5 Quantum Invariants of Periodic 3-Manifolds | 40 |
| Bibliography | 43 |
| Vita | 45 |

Abstract

In chapter 1, which represents joint work with Gilmer, we define an index two subcategory of a 3-dimensional cobordism category. The objects of the category are surfaces equipped with Lagrangian subspaces of their real first homology. This generalizes the result of [9] where surfaces are equipped with Lagrangian subspaces of their rational first homology. To define such subcategory, we give a formula for the parity of the Maslov index of a triple of Lagrangian subspaces of a skew symmetric bilinear form over \mathbb{R} .

In chapter 2, we find two bases for the lattices of the $SU(2)$ -TQFT-theory modules of the torus over given rings of integers. We find bases analogous to the bases defined in [13] for the lattices of the $SO(3)$ -TQFT-theory modules of the torus. Moreover, we discuss the quantization functors (V_p, Z_p) for $p = 1$, and $p = 2$. Then we give concrete bases for the lattices of the modules in the 2-theory. We use the above results to discuss the ideal invariant defined in [7]. The ideal can be computed for all the 3-manifolds using the 2-theory, and for all 3-manifolds with torus boundary using the $SU(2)$ -TQFT-theory. In fact, we show that this ideal using the $SU(2)$ -TQFT-theory is contained in the product of the ideals using the 2-theory and the $SO(3)$ -TQFT-theory under a certain change of coefficients, and it is equal in the case of torus boundary.

In chapter 3, we give a congruence which relates the quantum invariant of a prime-periodic 3-manifold to the quantum invariant of its orbit space. We do this for quantum invariant that is associated to any modular category over an integrally closed ground ring.

Chapter 1

The Parity of the Maslov Index and the Even Cobordism Category¹

1.1 Introduction

In [9], Gilmer considered a cobordism category \mathcal{C} . This category can be described roughly as follows. The objects of \mathcal{C} are closed surfaces equipped with Lagrangian subspaces of their rational first homology. A morphism of \mathcal{C} $N : \Sigma \rightarrow \Sigma'$ is a cobordism between Σ and Σ' . Also, he defined a subcategory \mathcal{C}^+ of \mathcal{C} of index two. It would be more consistent with other work [22] [23] to consider a similarly defined cobordism category \mathcal{C} where the extra data of a Lagrangian subspace is a subspace of the real first homology. The main goal of this chapter is define an analogous index two subcategory \mathcal{C}^+ of \mathcal{C} . We call \mathcal{C}^+ the even cobordism category. If one restricts to this ‘index two’ cobordism subcategory, one may obtain functors, related to the TQFT functors defined by Turaev with initial data a modular category, but without taking a quadratic extension of the ground ring of the modular category as is sometimes needed in [22, p.76].

It is not possible to simply modify the proof given in [9] for the existence of \mathcal{C}^+ to obtain a proof for the existence of \mathcal{C}^+ . This is because not every real Lagrangian subspace can be realized as the kernel of the map induced on first homology by the inclusion of a surface to a 3-manifold which has the surface as its boundary. Only the subspaces which are completions of subspaces of the rational homology can be so

¹This chapter incorporates material which appear in a paper with the same title which was coauthored with P. Gilmer; Fund. Math. Vol **184** (2005), 95-102.

realized. So another approach has to be used. We actually reduce the problem to the one already solved for \mathcal{C} but this requires some new algebraic results. These algebraic results may be of independent interest.

We prove the algebraic results in §1.2. This section is written without any appeal to topology. We derive the following congruence for the Maslov index, denoted μ :

Theorem 1.1.1 *Let V be a symplectic vector space and λ_1, λ_2 , and λ_3 be any three Lagrangian subspaces, then we have*

$$\begin{aligned}\mu(\lambda_1, \lambda_2, \lambda_3) &\equiv \dim(\lambda_1) + \sum_{1 \leq i < j \leq 3} \dim(\lambda_i \cap \lambda_j) \pmod{2} \\ &\equiv \dim(\lambda_1) + \sum_{1 \leq i < j \leq 3} \dim(\lambda_i + \lambda_j) \pmod{2}\end{aligned}$$

If $\lambda_1 \cap \lambda_2 = \lambda_2 \cap \lambda_3 = \lambda_1 \cap \lambda_3 = 0$, this result follows from [17, 1.5.7] which gives a formula for the Maslov index in terms of a special form these Lagrangian subspaces must take in this case. We give a very different proof. Theorem 1.1.1 will be the key to proving that the morphisms of C^+ are closed under composition. In §1.3, we describe the weighted cobordism categories \mathcal{C} and C in greater detail. In §1.4, C^+ is defined.

1.2 Lagrangian Subspaces and the Maslov Index

Let V be a symplectic vector space, i.e. V is finite dimensional over \mathbb{R} and endowed with a skew symmetric bilinear form ψ . This is the terminology used in [22]. Note that we do not require that the form is nondegenerate. If A is a subspace of V , its annihilator, $\text{Ann}(A)$, is the set of elements which pair under the form with all of A to give zero. If A and A' are two subspaces, then [22, IV.3.1.a, IV.3.1.1]

$$\text{Ann}(A + A') = \text{Ann}(A) \cap \text{Ann}(A') \tag{1.1}$$

$$\text{Ann}(A \cap A') = \text{Ann}(A) + \text{Ann}(A') \tag{1.2}$$

A subspace $A \subset V$ is said to be a Lagrangian subspace if $A = \text{Ann}(A)$. The proof of Theorem (1.1.1) is given in this section after we give all the results needed in the proof.

Theorem 1.2.1 [14] *Let (V, ψ) be a symplectic vector space and λ_1, λ_2 , and λ_3 be three Lagrangian subspaces. Then we have*

$$\dim(\lambda_1 + \lambda_2 + \lambda_3) \equiv \dim(\lambda_1 \cap \lambda_2 \cap \lambda_3) \pmod{2}$$

Proof. We have a skew symmetric bilinear form ψ on V . Now define a form $\{, \}$ on $(\lambda_1 + \lambda_2 + \lambda_3)/(\lambda_1 \cap \lambda_2 \cap \lambda_3)$ by $\{a, b\} = \psi(a, b)$ where $a, b \in (\lambda_1 + \lambda_2 + \lambda_3)/(\lambda_1 \cap \lambda_2 \cap \lambda_3)$. To show that this new form is well-defined, let $a_1, a_2 \in (\lambda_1 + \lambda_2 + \lambda_3)$ such that $\bar{a}_1 = \bar{a}_2$, i.e. $a_1 - a_2 \in (\lambda_1 \cap \lambda_2 \cap \lambda_3)$. It follows $\psi(a_1 - a_2, b) = 0$ for all $b \in (\lambda_1 + \lambda_2 + \lambda_3)$, so $\psi(a_1, b) = \psi(a_2, b)$. Hence $\{a_1, b\} = \{a_2, b\}$ for all $b \in (\lambda_1 + \lambda_2 + \lambda_3)$ that $\{, \}$ is well-defined. Since ψ is skew symmetric bilinear form, so is $\{, \}$. We now wish to show that $\{, \}$ is non-degenerate. Let $a \in (\lambda_1 + \lambda_2 + \lambda_3)/(\lambda_1 \cap \lambda_2 \cap \lambda_3)$ such that $\{a, b\} = 0$ for all $b \in (\lambda_1 + \lambda_2 + \lambda_3)/(\lambda_1 \cap \lambda_2 \cap \lambda_3)$, i.e. $\psi(a, b) = 0$ for all $b \in \lambda_1 + \lambda_2 + \lambda_3$, it implies that $a \in \text{Ann}(\lambda_1 + \lambda_2 + \lambda_3)$. By equation (1.1)

$$\begin{aligned} \text{Ann}(\lambda_1 + \lambda_2 + \lambda_3) &= \text{Ann}(\lambda_1 + \lambda_2) \cap \text{Ann}(\lambda_3) \\ &= (\text{Ann}(\lambda_1) \cap \text{Ann}(\lambda_2)) \cap \lambda_3 \\ &= \lambda_1 \cap \lambda_2 \cap \lambda_3. \end{aligned}$$

So $a \in \lambda_1 \cap \lambda_2 \cap \lambda_3$, i.e. $a = 0$ in $(\lambda_1 + \lambda_2 + \lambda_3)/(\lambda_1 \cap \lambda_2 \cap \lambda_3)$. As is well-known, a non-degenerate symplectic vector space must be even dimensional. Hence $(\lambda_1 + \lambda_2 + \lambda_3)/(\lambda_1 \cap \lambda_2 \cap \lambda_3)$ is of even dimension, so we get

$$\dim(\lambda_1 + \lambda_2 + \lambda_3) \equiv \dim(\lambda_1 \cap \lambda_2 \cap \lambda_3) \pmod{2}$$

■

We have the following well-known proposition [22, IV.3.5]

Proposition 1.2.1 *Let λ_1, λ_2 and λ_3 be three Lagrangian subspaces of V . Define a bilinear form \langle, \rangle on $(\lambda_1 + \lambda_2) \cap \lambda_3$ by*

$$\langle a, b \rangle = \psi(a_2, b) \tag{1.3}$$

where $a, b \in (\lambda_1 + \lambda_2) \cap \lambda_3$ and $a = a_1 + a_2$. \langle, \rangle is a well-defined symmetric bilinear form.

Proof. To show $\langle \cdot, \cdot \rangle$ is well-defined, note that the decomposition $a = a_1 + a_2$, where $a_1 \in \lambda_1$ and $a_2 \in \lambda_2$, is unique up to an element in $\lambda_1 \cap \lambda_2$, and this element annihilates b for all $b \in \lambda_1 + \lambda_2$. So the form is well-defined. As ψ is bilinear, $\langle \cdot, \cdot \rangle$ is bilinear. Let a be as before and $b = b_1 + b_2$ where $b_1 \in \lambda_1, b_2 \in \lambda_2$ and $b \in \lambda_3$. Since $\lambda_i = \text{Ann}(\lambda_i)$ for $i = 1, 2, 3$ and ψ is skew symmetric, we have

$$\begin{aligned}\psi(a_2, b) &= \psi(a - a_1, b) = \psi(a, b) - \psi(a_1, b) = \psi(b, a_1) = \psi(b_1 + b_2, a_1) \\ &= \psi(b_1, a_1) + \psi(b_2, a_1) + \psi(b_2, a_2) = \psi(b_2, a).\end{aligned}$$

Hence the form is symmetric. ■

Definition 1.2.1 *The Maslov index $\mu(\lambda_1, \lambda_2, \lambda_3)$ of the triple $(\lambda_1, \lambda_2, \lambda_3)$ is the signature of the form $\langle \cdot, \cdot \rangle$ defined above.*

In general, $\langle \cdot, \cdot \rangle$ is degenerate. In fact, it is known that its annihilator contains $(\lambda_1 \cap \lambda_3) + (\lambda_2 \cap \lambda_3)$ [22, p.182-183]. If $\lambda_1 \cap \lambda_2 = 0$, it is known that the annihilator is $(\lambda_1 \cap \lambda_3) + (\lambda_2 \cap \lambda_3)$ [17, 1.5.6]. It is shown in [14] that this is true in general.

Theorem 1.2.2 [14] *Let (V, ψ) be a symplectic vector space and λ_1, λ_2 , and λ_3 be three Lagrangian subspaces, then the induced form $\langle \cdot, \cdot \rangle$ on $(\lambda_1 + \lambda_2) \cap \lambda_3$ given in (1.3) has annihilator equal to $(\lambda_1 \cap \lambda_3) + (\lambda_2 \cap \lambda_3)$.*

Proof. Let W denote the annihilator of this form. It is clear that $\lambda_1 \cap \lambda_3 \subset W$, also $\lambda_2 \cap \lambda_3 \subset W$. Hence $(\lambda_1 \cap \lambda_3) + (\lambda_2 \cap \lambda_3) \subset W$. Now to prove the other containment, let $a \in W$, so $\langle a, b \rangle = 0$ for all $b \in (\lambda_1 + \lambda_2) \cap \lambda_3$. In other words; if $a = a_1 + a_2 \in \lambda_3$ where $a_1 \in \lambda_1$ and $a_2 \in \lambda_2$, then we have $\psi(a_2, b) = 0$. It follows that $a_2 \in \text{Ann}((\lambda_1 + \lambda_2) \cap \lambda_3)$ in V . Using equations (1.1) and (1.2), we have that

$$\begin{aligned}\text{Ann}((\lambda_1 + \lambda_2) \cap \lambda_3) &= \text{Ann}(\lambda_1 + \lambda_2) + \text{Ann}(\lambda_3) \\ &= (\text{Ann}(\lambda_1) \cap \text{Ann}(\lambda_2)) + \lambda_3 \\ &= (\lambda_1 \cap \lambda_2) + \lambda_3\end{aligned}$$

Thus $a_2 \in (\lambda_1 \cap \lambda_2) + \lambda_3$. So we could write $a_2 = c + d$ where $c \in \lambda_1 \cap \lambda_2$ and $d \in \lambda_3$. It follows that $a = (a_1 + c) + d$ where $a_1 + c \in \lambda_1$ and $d \in \lambda_3$. Now

since we have $a_2, c \in \lambda_2$ we get $d \in \lambda_2$. Since $a, d \in \lambda_3$ we get $a_1 + c \in \lambda_3$. Hence $d \in \lambda_2 \cap \lambda_3$ and $a_1 + c \in \lambda_1 \cap \lambda_3$. So $a = (a_1 + c) + d \in (\lambda_1 \cap \lambda_3) + (\lambda_2 \cap \lambda_3)$. Thus $W \subset (\lambda_1 \cap \lambda_3) + (\lambda_2 \cap \lambda_3)$. So $W = (\lambda_1 \cap \lambda_3) + (\lambda_2 \cap \lambda_3)$, i.e. the annihilator of the form \langle, \rangle is equal to $(\lambda_1 \cap \lambda_3) + (\lambda_2 \cap \lambda_3)$. ■

Proposition 1.2.2 *For any pair of Lagrangian subspaces λ_1 , and λ_2 we have*

$$\dim(\lambda_1) = \dim(\lambda_2)$$

and

$$\dim(\lambda_1 + \lambda_2) \equiv \dim(\lambda_1 \cap \lambda_2) \pmod{2}. \quad (1.4)$$

Proof. The first formula follows by reducing it to the nonsingular case and

$$\dim(A) = \dim(V) - \dim(\text{Ann}(A)).$$

We obtain the second congruence from

$$\dim(A + B) = \dim(A) + \dim(B) - \dim(A \cap B) \quad (1.5)$$

and the first formula. ■

Corollary 1.2.1

$$\mu(\lambda_1, \lambda_2, \lambda_3) \equiv \dim((\lambda_1 + \lambda_2) \cap \lambda_3) + \dim((\lambda_1 \cap \lambda_3) + (\lambda_2 \cap \lambda_3)) \pmod{2}.$$

Proof. Since the annihilator of the form is $(\lambda_1 \cap \lambda_3) + (\lambda_2 \cap \lambda_3)$, it follows that the rank of the form is

$$\dim((\lambda_1 + \lambda_2) \cap \lambda_3) - \dim((\lambda_1 \cap \lambda_3) + (\lambda_2 \cap \lambda_3)).$$

The result follows as the signature and the rank of a nondegenerate form agree modulo two. ■

Proof of Theorem 1.1.1. By equation (1.5), we have

$$\begin{aligned} \dim(\lambda_1 + \lambda_2 + \lambda_3) &\equiv \dim(\lambda_1) + \dim(\lambda_2 + \lambda_3) + \\ &\quad \dim(\lambda_1 \cap (\lambda_2 + \lambda_3)) \pmod{2} \end{aligned}$$

and also have

$$\begin{aligned} \dim(\lambda_1 \cap \lambda_2 \cap \lambda_3) &\equiv \dim(\lambda_1 \cap \lambda_2) + \dim(\lambda_1 \cap \lambda_3) + \\ &\quad \dim((\lambda_1 \cap \lambda_2) + (\lambda_1 \cap \lambda_3)) \pmod{2}. \end{aligned}$$

Hence by Theorem (1.2.1), the left hand sides of these two congruences are congruent. So their right hand sides must be congruent as well:

$$\begin{aligned} \dim(\lambda_1) + \dim(\lambda_2 + \lambda_3) + \dim(\lambda_1 \cap (\lambda_2 + \lambda_3)) &\equiv \dim(\lambda_1 \cap \lambda_2) + \dim(\lambda_1 \cap \lambda_3) \\ &\quad + \dim((\lambda_1 \cap \lambda_2) + (\lambda_1 \cap \lambda_3)) \pmod{2}. \end{aligned}$$

The last equation is equivalent to

$$\begin{aligned} \dim(\lambda_1 \cap (\lambda_2 + \lambda_3)) + \dim((\lambda_1 \cap \lambda_2) + (\lambda_1 \cap \lambda_3)) &\equiv \dim(\lambda_1) + \\ \dim(\lambda_2 + \lambda_3) + \dim(\lambda_1 \cap \lambda_2) + \dim(\lambda_1 \cap \lambda_3) &\pmod{2}. \end{aligned}$$

The left hand side of this last equation is congruent to the Maslov index by Corollary 1.2.1, and hence the first formula follows. The second formula follows by equation (1.4). ■

The following results from [22] give some properties of the Maslov index and will be used in the next section.

Lemma 1.2.1 *For any Lagrangian subspaces $\lambda_1, \lambda_2, \lambda_3$ of V , we have*

$$\mu(\lambda_1, \lambda_2, \lambda_3) = -\mu(\lambda_2, \lambda_1, \lambda_3) = -\mu(\lambda_1, \lambda_3, \lambda_2)$$

Lemma 1.2.2 *For any Lagrangian subspaces $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of V , we have*

$$\mu(\lambda_1, \lambda_2, \lambda_3) - \mu(\lambda_1, \lambda_2, \lambda_4) + \mu(\lambda_1, \lambda_3, \lambda_4) - \mu(\lambda_2, \lambda_3, \lambda_4) = 0$$

Definition 1.2.2 *Let V, V' be symplectic vector spaces. The symplectic vector space $-V$ is just V as a vector space with the opposite (minus) form. Also, the symplectic vector space $-V \oplus V'$ is the direct sum of V and V' as a vector space with the sum of the two forms of $-V$ and V' .*

Definition 1.2.3 Let V, V' be symplectic vector spaces. Let $\chi \subset -V \oplus V'$, $\lambda \subset -V$, and $\lambda' \subset V'$ be Lagrangian subspaces. Then we define the Lagrangian $\chi_*(\lambda) = \{y \in V' \mid \text{such that } (x, y) \in \chi \text{ for some } x \in \lambda\}$, and the Lagrangian $\chi^*(\lambda') = \{x \in V \mid \text{such that } (x, y) \in \chi \text{ for some } y \in \lambda'\}$.

Lemma 1.2.3 Let V, V' be non-degenerate symplectic vector spaces and let $\chi \subset -V \oplus V'$ be a Lagrangian subspace. Let $\lambda_1, \lambda_2 \subset -V$ and $\lambda'_1, \lambda'_2 \subset V'$ be Lagrangian subspaces. Then

$$\mu(\lambda_1, \lambda_2, \chi^*(\lambda'_1)) + \mu(\chi_*(\lambda_1), \lambda'_1, \lambda'_2) = \mu(\lambda_1, \lambda_2, \chi^*(\lambda'_2)) + \mu(\chi_*(\lambda_2), \lambda'_1, \lambda'_2).$$

1.3 The Weighted Cobordism Categories

All 3-manifolds and surfaces in this chapter are assumed to be oriented and compact. We define a weighted cobordism category \mathcal{C} whose objects are surfaces Σ without boundary equipped with a Lagrangian subspace $\lambda \subset H_1(\Sigma, \mathbb{R})$. We will denote objects by pairs (Σ, λ) . A cobordism from (Σ, λ) to (Σ', λ') is a 3-manifold together with an orientation preserving homeomorphism (called its boundary identification) from its boundary to $-\Sigma \sqcup \Sigma'$. Here, and elsewhere, $-\Sigma$ denotes Σ with the opposite orientation. Two cobordisms are equivalent if there is an orientation preserving homeomorphism between the underlying 3-manifolds that commutes with the boundary identifications. A morphism $M : (\Sigma, \lambda) \rightarrow (\Sigma', \lambda')$ is an equivalence class of cobordisms from (Σ, λ) to (Σ', λ') together with an integer weight. We denote morphisms by a single letter. We let $w(M)$ denote the weight of M . We let $\mathfrak{b}M$ denote the underlying 3-manifold of a representative cobordism. This is well defined up to homeomorphism respecting the boundary identifications. We call (Σ, λ) the source of M and (Σ', λ') the target of M . We let j_M denote the inclusion Σ into $\mathfrak{b}M$, and j^M denote the inclusion Σ' into $\mathfrak{b}M$. Here and sometimes below we ignore the boundary identifications for simplicity and we write as if $\Sigma \sqcup \Sigma'$ were the boundary of $\mathfrak{b}M$.

If $M : (\Sigma, \lambda) \rightarrow (\Sigma', \lambda')$ is a cobordism, then let $j_{M*} : H_1(\Sigma, \mathbb{R}) \rightarrow H_1(M, \mathbb{R})$ and $j_*^M : H_1(\Sigma', \mathbb{R}) \rightarrow H_1(M, \mathbb{R})$ be the maps induced by the inclusions of Σ and Σ' in M respectively. We define $k(M) = \ker(j_{M*} \amalg j_*^M) \in -H_1(\Sigma, \mathbb{R}) \oplus H_1(\Sigma', \mathbb{R})$. Also,

we let $M_*(\lambda)$ denote the Lagrangian subspace [22, p188-189] of $H_1(\Sigma', \mathbb{R})$ given by $(j_*^M)^{-1}(j_{M_*}(\lambda))$. Similarly we have the Lagrangian subspace of $H_1(\Sigma, \mathbb{R})$: $M^*(\lambda') = (j_{M_*})^{-1}(j_*^M(\lambda'))$. From the definition of these last two Lagrangian subspaces, we obtain that $M_*(\lambda) = k(M)_*(\lambda)$, and $M^*(\lambda') = k(M)^*(\lambda')$.

If $M_1 : (\Sigma, \lambda) \rightarrow (\Sigma', \lambda')$ and $M_2 : (\Sigma', \lambda') \rightarrow (\Sigma', \lambda'')$ are two morphisms we define $\flat(M_2 \circ M_1)$ by gluing $\flat M_2$ to $\flat M_1$ by identifying the target of M_1 to the source of M_2 . The boundary of this new 3-manifold is equipped with a boundary identification in the obvious way. The weight of the composition is given by the formula ¹.

$$w(M_2 \circ M_1) = w(M_1) + w(M_2) - \mu(M_{1*}(\lambda), \lambda', M_2^*(\lambda'')) \quad (1.6)$$

The identity $\text{id}_{(\Sigma, \lambda)} : (\Sigma, \lambda) \rightarrow (\Sigma, \lambda)$ is given by $\Sigma \times I$ with the weight zero and the standard boundary identification. This is called a cylinder. Any morphism $C : (\Sigma, \lambda) \rightarrow (\Sigma, \lambda')$ with $\Sigma \times I$ as the underlying 3-manifold, and with the standard boundary identification will be called a skew-cylinder over Σ .

Lemma 1.3.1 *Skew-cylinders are invertible in \mathcal{C} . The inverse of $C : (\Sigma, \lambda) \rightarrow (\Sigma, \lambda')$ is the skew-cylinder from (Σ, λ') to (Σ, λ) with weight $-w(C)$.*

Proof. This follows immediately from the definitions. One needs that the Maslov index vanishes when two of the three Lagrangian subspaces coincides which follows by [22, p183], or Theorem (1.2.2). ■

If we make the same definitions but using Lagrangian subspaces in $H_1(\Sigma, \mathbb{Q})$, we obtain the cobordism category \mathcal{C} studied in [9]. The proof of the following result is closely related to the proof of [22, Lem. 9.1.1].

Lemma 1.3.2 *\mathcal{C} and \mathcal{C} are categories.*

Proof. To prove that \mathcal{C} or \mathcal{C} is a category, we need to satisfy the two category axioms. For the first one, we claim that the cylinder C over (Σ, λ) with weight zero is the identity morphism for the object (Σ, λ) . To prove the claim, let M be any morphism

¹As in [9], we adopt the sign convention of [23] rather than [22] for the sign of the Maslov index term in this formula. It makes no real difference for this chapter.

from (Σ, λ) to (Σ', λ') . Then $M \circ C$ is homeomorphic to M , and

$$\begin{aligned} w(M \circ C) &= w(M) + w(C) - \mu(C_*(\lambda), \lambda, M^*(\lambda')) \\ &= w(M) - \mu(\lambda, \lambda, M^*(\lambda')) \\ &= w(M) \end{aligned}$$

The last equality follows as the form is zero by Theorem (1.2.2). Hence we obtain that the first axiom holds. We claim that the composition is associative which is the second axiom. To prove this claim, we let $(M_i, w_i) : (\Sigma_{i-1}, \chi_{i-1}) \rightarrow (\Sigma_i, \chi_i)$ for $i = 1, 2, 3$. We know that $M_3 \circ (M_2 \circ M_1)$ is homeomorphic as a 3-manifold to $(M_3 \circ M_2) \circ M_1$, so it is left to prove that they have the same weight, i.e. $w(M_3 \circ (M_2 \circ M_1)) = w((M_3 \circ M_2) \circ M_1)$. We apply formula (1.6) to obtain

$$\begin{aligned} w(M_3 \circ (M_2 \circ M_1)) &= w(M_3) + w(M_2 \circ M_1) - \mu((M_2 \circ M_1)_*(\chi_0), \chi_2, M_3^*(\chi_3)) \\ &= w(M_3) + w(M_2) + w(M_1) - \mu((M_2 \circ M_1)_*(\chi_0), \chi_2, M_3^*(\chi_3)) \\ &\quad - \mu(M_{1*}(\chi_0), \chi_1, M_2^*(\chi_2)). \end{aligned}$$

Similarly;

$$\begin{aligned} w((M_3 \circ M_2) \circ M_1) &= w(M_3 \circ M_2) + w(M_1) - \mu(M_{1*}(\chi_0), \chi_1, (M_3 \circ M_2)^*(\chi_3)) \\ &= w(M_3) + w(M_2) + w(M_1) - \mu(M_{1*}(\chi_0), \chi_1, (M_3 \circ M_2)^*(\chi_3)) \\ &\quad - \mu(M_{2*}(\chi_1), \chi_2, M_3^*(\chi_3)). \end{aligned}$$

So if we compare the above two equations, then we need to prove that

$$\begin{aligned} \mu((M_2 \circ M_1)_*(\chi_0), \chi_2, M_3^*(\chi_3)) + \mu(M_{1*}(\chi_0), \chi_1, M_2^*(\chi_2)) &= \\ \mu(M_{1*}(\chi_0), \chi_1, (M_3 \circ M_2)^*(\chi_3)) + \mu(M_{2*}(\chi_1), \chi_2, M_3^*(\chi_3)). \end{aligned}$$

This follows from Lemma (1.2.3) and Lemma (1.2.1) using the facts [22, pp. 182]

$$(M_3 \circ M_2)^*(\chi_3) = M_2^*(M_3^*(\chi_3)),$$

and

$$(M_2 \circ M_1)_*(\chi_0) = M_{2*}(M_{1*}(\chi_0)),$$

in the case

$$\lambda_1 = \chi_1, \lambda'_1 = \chi_2, \lambda_2 = M_{1*}(\chi_0), \lambda'_2 = M_3^*(\chi_3), \text{ and}$$

$$N = k(M_2).$$

Hence we obtain that the second axiom holds. Therefore we conclude that \mathcal{C} and \mathbf{C} are cobordism categories. ■

As $H_1(\Sigma, \mathbb{Q}) \otimes \mathbb{R}$ is naturally isomorphic to $H_1(\Sigma, \mathbb{R})$, a Lagrangian in $H_1(\Sigma, \mathbb{Q})$ determines one in $H_1(\Sigma, \mathbb{R})$. A Lagrangian of $H_1(\Sigma, \mathbb{R})$ which arises in this way is called rational. In this way, we obtain a functor $\mathcal{C} \rightarrow \mathbf{C}$.

1.4 The Even Cobordism Category

We repeat a definition from [9] except now we apply it to morphisms of \mathbf{C} instead of \mathcal{C} . We denote $\beta_i(bM)$ by $\beta_i(M)$.

Definition 1.4.1 *A cobordism $M : (\Sigma, \lambda) \rightarrow (\Sigma', \lambda')$ of \mathbf{C} is even if and only if*

$$w(M) \equiv \dim(j_{M*}(\lambda) + j_*^M(\lambda')) + \beta_1(M) + \beta_0(M) + \beta_0(\Sigma) + \frac{\beta_1(\Sigma')}{2} + \epsilon(M) \pmod{2}$$

where $\epsilon(M)$ is one if exactly one of Σ and Σ' is nonempty and otherwise $\epsilon(M)$ is zero.

If a cobordism is not even, it is called odd.

We note that the inverse of an even skew-cylinder is even.

The first author showed that the composite of two even morphisms of \mathcal{C} is again even [9, Theorem 7.2]. The subcategory \mathcal{C}^+ was defined to be the category with the same objects as \mathcal{C} but with only even morphisms. In the rest of this section, we generalize this result to morphisms in \mathbf{C} . Given this result, we define the subcategory \mathbf{C}^+ to be the category with the same objects as \mathbf{C} but with only the even morphisms. We would also get a subcategory if we left the $\epsilon(N)$ term out of Definition 1.4.1. However the definition that we give is more natural from some points of view [9].

Proposition 1.4.1 *A skew-cylinder $C : (\Sigma, \lambda) \rightarrow (\Sigma, \lambda')$ is even if and only if*

$$w(C) \equiv \frac{\beta_1(\Sigma)}{2} + \dim(\lambda + \lambda') \pmod{2}$$

Proof. Apply the definition above. ■

Lemma 1.4.1 *Let $M : (\Sigma, \lambda) \rightarrow (\Sigma', \lambda')$ be an even morphism. If $C : (\Sigma, \hat{\lambda}) \rightarrow (\Sigma, \lambda)$ and $C' : (\Sigma', \lambda') \rightarrow (\Sigma', \tilde{\lambda})$ are even skew-cylinders, then $M \circ C$ and $C' \circ M$ are even.*

Proof. We first show that $M \circ C$ is even. We need to show

$$w(M \circ C) \equiv \dim(j_{M*}(\hat{\lambda}) + j_*^M(\lambda')) + \beta_1(M) + \beta_0(M) + \beta_0(\Sigma) + \frac{\beta_1(\Sigma_1)}{2} + \epsilon(M) \pmod{2} \quad (1.7)$$

By Equation (1.6),

$$w(M \circ C) \equiv w(M) + w(C) + \mu(\hat{\lambda}, \lambda, M^*(\lambda')) \pmod{2} \quad (1.8)$$

By assumption, we have that:

$$w(M) \equiv \dim(j_{M*}(\lambda) + j_*^M(\lambda')) + \beta_1(M) + \beta_0(M) + \beta_0(\Sigma) + \frac{\beta_1(\Sigma_1)}{2} + \epsilon(M) \pmod{2} \quad (1.9)$$

and,

$$w(C) \equiv \frac{\beta_1(\Sigma)}{2} + \dim(\hat{\lambda} + \lambda) \pmod{2}. \quad (1.10)$$

So after we substitute (1.8), (1.9) and (1.10) into (1.7), we conclude that we need only prove:

$$\begin{aligned} \dim(j_{M*}(\hat{\lambda}) + j_*^M(\lambda')) + \mu(\hat{\lambda}, \lambda, M^*(\lambda')) + \dim(\hat{\lambda} + \lambda) + \frac{\beta_1(\Sigma)}{2} \equiv \\ \dim(j_{M*}(\lambda) + j_*^M(\lambda')) \pmod{2} \end{aligned}$$

Given Theorem 1.1.1, this last congruence becomes:

$$\begin{aligned} \dim(j_{M*}(\hat{\lambda}) + j_*^M(\lambda')) + \dim(\hat{\lambda} + M^*(\lambda')) \equiv \\ \dim(j_{M*}(\lambda) + j_*^M(\lambda')) + \dim(\lambda + M^*(\lambda')) \pmod{2} \end{aligned} \quad (1.11)$$

For any subspace δ of $H_1(\Sigma, \mathbb{R})$, we have that

$$\delta + M^*(\lambda') = (j_{M*})^{-1} (j_{M*}(\delta) + j_*^M(\lambda'))$$

as

$$j_{M*}(\delta + M^*(\lambda')) = j_{M*}(\delta) + j_*^M(\lambda')$$

and kernel of j_{M*} is a subset of $M^*(\lambda')$. Thus we have that $\dim(\delta + M^*(\lambda')) = \dim(j_{M*}(\delta) + j_*^M(\lambda')) + n$ where n is the dimension of kernel of j_{M*} . Thus both sides of (1.11) are congruent to n . Hence, we obtain (1.7).

The proof that $C' \circ M$ is even follows formally from the first part, if we consider how the parity of a cobordism changes when we reverse the orientation of the underlying 3-manifold and reverse the roles of source and target. ■

Proposition 1.4.2 *If there are even skew-cylinders C and C' over Σ , and Σ' such that $C \circ M \circ C'$ is even, then M is an even cobordism in C from (Σ, λ) to (Σ', λ') .*

Proof. It follows by lemma (1.3.1) that we can factor M as $C^{-1} \circ C \circ M \circ C' \circ C'^{-1}$. Hence M is even by two applications of Lemma (1.4.1). ■

Theorem 1.4.1 *The composition of two even morphisms of C is again even.*

Proof. Let M_1, M_2 be two even morphisms and adopt the notations associated to M_1 and M_2 in §1.3. We need to show that $M_2 \circ M_1$ is an even cobordism. It suffices to show $C'' \circ M_2 \circ M_1 \circ C$ is even for some even skew-cylinders over C and C'' over Σ and Σ'' with rational Lagrangian subspaces for Σ and Σ'' . On the other hand we can write $M_2 \circ M_1$ as $M_2 \circ C' \circ C'^{-1} \circ M_1$ where C' is an even skew-cylinder over Σ' whose the target has a rational Lagrangian. We have that

$$C'' \circ M_2 \circ M_1 \circ C = C'' \circ M_2 \circ C' \circ C'^{-1} \circ M_1 \circ C = N_2 \circ N_1$$

where $N_2 = C'' \circ M_2 \circ C'$ and $N_1 = C'^{-1} \circ M_1 \circ C$. By Lemma 1.4.1, N_1, N_2 are even morphisms. By Theorem 7.2 in [9], it follows that $N_2 \circ N_1$ is even. Hence $M_2 \circ M_1$ is even. ■

Chapter 2

Integral Bases for Certain TQFT Modules of the Torus

2.1 Introduction

We let p denote an odd prime or twice an odd prime unless mentioned otherwise. Also, we let Σ denote a surface of genus g . Gilmer defined an integral TQFT-functor \mathcal{S}_p in [9] based on the integrality results of the $SO(3)$ - and $SU(2)$ -invariants in [19, 18]. This is a functor that associates to a closed surface Σ , a module $\mathcal{S}_p(\Sigma)$ over a certain cyclotomic ring of integers \mathcal{O}_p . Moreover, Gilmer showed that these modules are free in the case of p is an odd prime. Gilmer and Masbaum constructed basis for $\mathcal{S}_p(\Sigma)$ and gave an independent proof of freeness in this case. In addition, Gilmer showed that these modules are projective where p is twice an odd prime. In this chapter, we prove that the modules $\mathcal{S}_p(S^1 \times S^1)$ are free by constructing two explicit bases in the case p is twice an odd prime. In the 2-theory, we prove also that the modules $\mathcal{S}_2(\Sigma)$ are free by constructing an explicit basis for any surface.

Frohman and Kania-Bartoszyńska in [7] defined an ideal invariant of 3-manifolds with boundary using the $SU(2)$ -TQFT-theory that is hard to compute. In fact, they make use of another ideal that they defined to give an estimate for this ideal. However, Gilmer and Masbaum in [12] computed an analogous ideal invariant using the $SO(3)$ -TQFT-theory for 3-manifolds that are obtained by doing surgery along a knot in the complement of another knot. The computations depend entirely on the fact that bases are constructed for the integral lattices of the $SO(3)$ -TQFT-theory modules [13, 12] of the torus. Also, Gilmer and Masbaum gave a finite set of generators

for this ideal in general. Based on our results in this chapter, we compute this ideal for the above 3-manifolds with torus boundary using the $SU(2)$ -TQFT-theory. Also, we introduce a formula to give an estimate for the ideal using the $SU(2)$ -TQFT-theory in terms of the ideals using the 2- and $SO(3)$ -TQFT-theories. In fact, the same formula can be used to compute this ideal using the $SU(2)$ -TQFT-theory for the all the above 3-manifolds with torus boundary.

In §2.1, we describe the $SO(3)$ - and $SU(2)$ -TQFT-functors using the approach of [3] over a variant ring depending on p . We review the integral TQFT-functors in §2.2 that Gilmer defined in [9]. The first bases for the lattices of the $SU(2)$ -TQFT-modules are given in §2.3. We review the Frohman Kania-Bartoszyńska ideal in §2.4, and then we draw some conclusions based on the results of the previous section regarding this ideal. The quantization functors for $p = 1$ and $p = 2$ are discussed in §2.5, again following [3]. Also in this section, we give basis for $\mathcal{S}_2(\Sigma)$, and then draw some conclusions regarding the Frohman and Kania-Bartoszyńska ideal for this theory. We reformulate some results given in [3] in §2.6 to serve our need. Finally, we give another bases for the lattices of the $SU(2)$ -TQFT-modules in §2.7. The advantage of this one over the first basis is that it allows us to prove Theorem (2.8.3).

2.2 The $SO(3)$ - and $SU(2)$ -TQFTs

We consider the (2+1)-dimensional TQFT constructed as the main example of [3, P. 456] with some modifications. In particular, we use the cobordism category \mathcal{C} discussed in [9, 14] where the 3-manifolds have banded links but surfaces do not have colored points. Hence the objects are oriented surfaces with extra structure (Lagrangian subspaces of their first real homology). The cobordisms are equivalence classes of compact oriented 3-manifolds with extra structure (an integer weight) with banded links sitting inside of them. Two cobordisms with the same weight are said to be equivalent if there is an orientation preserving diffeomorphism that fixes the boundary.

Let

$$k_p = \begin{cases} \mathbb{Z}[A_p, \frac{1}{p}], & \text{if } p \equiv -1 \pmod{4}; \\ \mathbb{Z}[\alpha_p, \frac{1}{p}], & \text{if } p \equiv 1 \text{ or } 2 \pmod{4}. \end{cases}$$

Here and elsewhere A_p, α_p are ζ_{2p} and ζ_{4p} respectively for $p \geq 3$.

Now, we consider the TQFT-functor (V_p, Z_p) from \mathcal{C} to the category of finitely

generated projective k_p -modules. The functor (V_p, Z_p) is defined as follows. $V_p(\Sigma)$ is a quotient of the free k_p -module generated by all cobordisms with boundary Σ , and $Z_p(M)$ is the k_p -linear map from $V_p(\Sigma)$ to $V_p(\Sigma')$ (where $\partial M = -\Sigma \amalg \Sigma'$) induced by gluing representatives of elements of $V_p(\Sigma)$ to M along Σ via the identification map of the first component of the boundary.

If M is a closed cobordism, then $Z_p(M)$ is the multiplication by the scalar $\langle M \rangle_p$ defined in [3, §.2]. This invariant is normalized in two other ways. The first normalization of this invariant is $I_p(M) = \mathcal{D}_p \langle M_b \rangle_p$. Here and elsewhere M_b is the 3-manifold M with a reassigned weight zero, and $\mathcal{D}_p = \langle S_b^3 \rangle_p^{-1}$. The second normalization is $\theta_p(M) = \mathcal{D}_p^{\beta_1(M)+1} \langle M_b \rangle_p$, i.e $\theta_p(M) = \mathcal{D}_p^{\beta_1(M)} I_p(M)$.

If $\partial M = \Sigma$ and M is considered as a cobordism from \emptyset to Σ , then $Z_p(M)(1) \in V_p(\Sigma)$ is denoted by $[M]_p$ and called a *vacuum state* and it is *connected* if M is connected. Finally, it is known that V_p is generated over k_p by all connected vacuum states.

The modules $V_p(\Sigma)$ are free modules over k_p , and carry a nonsingular Hermitian sesquilinear form

$$\langle \cdot, \cdot \rangle_\Sigma : V_p(\Sigma) \times V_p(\Sigma) \rightarrow k_p,$$

given by

$$\langle [M_1], [M_2] \rangle_\Sigma = \langle M_1 \cup_\Sigma -M_2 \rangle_p. \quad (2.1)$$

Here $-M$ is the cobordism M with the orientation reversed and multiplying the integer weight by -1, and leaving the Lagrangian subspace on the boundary the same.

Let $d_p = \lfloor \frac{p-1}{2} \rfloor$, it is known that d_p is the dimension of $V_p(S^1 \times S^1)$. One has that $V_p(S^1 \times S^1) \cong k_p[z]/I$ where the ideal I is generated by $e_{d_p} - e_{d_p-1}$ in the case of p is an odd prime and by e_{d_p} in the case of p is twice an odd prime (See [1] for more details). Thus indeed, $V_p(S^1 \times S^1)$ has a basis $\{e_0, \dots, e_{d_p-1}\}$ of rank d_p .

2.3 The Integral Cobordism Functor

Let \mathcal{C}' be the subcategory of \mathcal{C} consisting of the nonempty connected surfaces and connected cobordisms between them. Let \mathcal{O}_p be the ring of integers of the ring k_p defined before. The ring of integers is given by

$$\mathcal{O}_p = \begin{cases} \mathbb{Z}[A_p], & \text{if } p \equiv -1 \pmod{4}; \\ \mathbb{Z}[\alpha_p], & \text{if } p \equiv 1 \text{ or } 2 \pmod{4}. \end{cases}$$

Thus the ring of integers of k_p is a Dedekind domain.

Definition 2.3.1 *For the surface Σ , we define $\mathcal{S}_p(\Sigma)$ to be the \mathcal{O}_p -submodule of $V_p(\Sigma)$ generated by all connected vacuum states.*

If $M : \Sigma \rightarrow \Sigma'$ is a cobordism of \mathcal{C}' , then $Z_p(M)([N]_p) = [M \cup_{\Sigma} -N]_p \in \mathcal{S}_p(\Sigma')$. Hence we obtain a functor from \mathcal{C}' to the category of \mathcal{O}_p -modules. These modules are projective as they are finitely generated torsion-free over Dedekind domains [9, Thm. 2.5]. Also, these modules carry an \mathcal{O}_p -Hermitian sesquilinear form

$$(\ , \)_{\Sigma} : \mathcal{S}_p(\Sigma) \times \mathcal{S}_p(\Sigma) \rightarrow \mathcal{O}_p,$$

given by

$$([M_1], [M_2])_{\Sigma} = \mathcal{D}_p \langle [M_1], [M_2] \rangle = \mathcal{D}_p \langle M_1 \cup_{\Sigma} -M_2 \rangle_p, \quad (2.2)$$

The value of this form always lies in \mathcal{O}_p by the integrality results for closed 3-manifolds in [19, 18].

If $R \subseteq \mathcal{S}_p(\Sigma)$ is an \mathcal{O}_p -submodule define

$$R^{\sharp} = \{v \in V_p(\Sigma) \mid (r, v)_{\Sigma} \in \mathcal{O}_p, \forall r \in R\},$$

then we can conclude

$$R \subseteq \mathcal{S}_p(\Sigma) \subseteq \mathcal{S}_p^{\sharp}(\Sigma) \subseteq R^{\sharp}. \quad (2.3)$$

Definition 2.3.2 *A Hermitian sesquilinear form on a projective module over a Dedekind domain is called non-degenerate if the adjoint map is injective, and unimodular if the adjoint map is an isomorphism.*

For our use, if the matrix of the form has a nonzero (unit) determinant, then the form will be non-degenerate (unimodular) respectively. Note that the determinant of the form (2.2) is nonzero as the form (2.1) is non-degenerate. Hence the form (2.2) is non-degenerate. In fact, we prove that the form (2.2) is unimodular for the 2-theory (discussed in §5) for all surfaces and for $S^1 \times S^1$ in the case of p is twice an odd prime.

A standard basis $\{u_{\sigma}\}$ for $V_p(\Sigma)$ is given (see [3]) in terms of p -admissible colorings σ of the spine of a handlebody of genus g whose boundary is Σ where the set of colors is $\{0, 1, 2, \dots, d_p - 1\}$, and the sum of the colors at a 3-vertex is even and less than $2d_p$ in the case that p is twice an odd prime.

All of the above elements u_σ lie in $\mathcal{S}_p(\Sigma)$ when p is twice an odd prime. This follows as the quantum integers (denominators of the Jones-Wenzel idempotents) are units in \mathcal{O}_p (see Corollary 2.7.1). An admissible colored trivalent graph [3] is to be interpreted, here and elsewhere, as an \mathcal{O}_p -linear combination of links.

We say $a \sim b$ in \mathcal{O}_p if a/b is a unit in \mathcal{O}_p . The following proposition is an elementary fact from number theory that gives us a family of units in the ring \mathcal{O}_p .

Proposition 2.3.1 ([23]) *Suppose n has at least two distinct prime factors. Then $1 - \zeta_n$ is a unit in $\mathbb{Z}[\zeta_n]$.*

We make use of the following lemma in giving the first basis for $\mathcal{S}_p(S^1 \times S^1)$ in §3 whose proof will be in §2.7.

Lemma 2.3.1

$$\mathcal{D}_p \sim \begin{cases} (1 - A_p^2)^{\frac{p-3}{2}}, & \text{if } p \text{ is an odd prime;} \\ \sqrt{2}(1 + \alpha_p^4)^{\frac{p-3}{2}}, & \text{if } p \text{ is twice an odd prime.} \end{cases}$$

Proposition 2.3.2 *The elements $\{u_\sigma\}$ are orthogonal with respect to the form (2.2). Moreover, we have*

$$(u_\sigma, u_\sigma)_\Sigma \sim \mathcal{D}_p^g. \quad (2.4)$$

Proof. If we use Theorem (4.11) in [3], and the facts

$$\langle S^3 \rangle_p = \mathcal{D}_p^{-1}, \quad \sharp v - \sharp e = 1 - g.$$

We obtain the result from knowing that all the quantum integers are units over \mathcal{O}_p ([13, Lem. 4.1], Corollary(2.7.1)), and using the definition of $(\ , \)_\Sigma$. ■

Definition 2.3.3 *Let $[i]_p$ denote $\frac{A_p^{2i} - A_p^{-2i}}{A_p^2 - A_p^{-2}}$. This is called the i -th quantum integer.*

We can describe the modules $\mathcal{S}_p(\Sigma)$ in terms of ‘mixed graph’ notation in a fixed connected 3-manifold M whose boundary is Σ . By a mixed graph, we mean a p -admissibly trivalent graph whose simple closed curves may be colored ω_p or an integer from the set $\{0, 1, \dots, p-2\}$ where

$$\omega_p = \mathcal{D}_p^{-1} \sum_{i=0}^{d_p-1} (-1)^i [i+1]_p e_i.$$

Using the surgery axiom (S2) in [3], we can choose this fixed 3-manifold to be a handlebody whose boundary is Σ . Thus we have

Proposition 2.3.3 *A mixed graph in a connected 3-manifold with boundary Σ represents an element in $\mathcal{S}_p(\Sigma)$. Moreover, $\mathcal{S}_p(\Sigma)$ is generated over \mathcal{O}_p by all the elements given by a mixed graph in a fixed handlebody whose boundary is Σ with the same genus.*

Proof. The first statement follows from that fact that V_p satisfies the second surgery axiom. The second statement follows as every 3-manifold with boundary Σ is obtained by a sequence of 2-surgeries to a handlebody of the same boundary and the definition of $\mathcal{S}_p(\Sigma)$. ■

2.4 The First Basis for $\mathcal{S}_p(S^1 \times S^1)$

In this section, we assume r is an odd prime and $p = 2r$. We give a standard basis for $\mathcal{S}_p(S^1 \times S^1)$. We need the following lemma before we state our basis.

Definition 2.4.1 *Let μ_i be the eigenvalue for the eigenvector e_i of the twist map on the Kauffman skein module of the solid torus. It is known in [1] that $\mu_i = (-1)^i A^{i(i+2)}$.*

Lemma 2.4.1 *For $i \neq j$, we have $\mu_i - \mu_j$ is equivalent to one of the following three cases up to a unit in \mathcal{O}_p .*

1. 1 if $i \not\equiv j \pmod{2}$ and $(j-i)(i+j+2) \not\equiv 0 \pmod{r}$.
2. $\sqrt{2}$ if $i \not\equiv j \pmod{2}$ and $(j-i)(i+j+2) \equiv 0 \pmod{r}$.
3. $1 + \alpha_p^4$ if $i \equiv j \pmod{2}$.

Proof. Without loss of generality we can assume $0 \leq i < j \leq d_p - 1$. We have

$$\begin{aligned} \mu_i - \mu_j &= (-1)^i \alpha_p^{2i(i+2)} - (-1)^j \alpha_p^{2j(j+2)} \\ &= (-1)^i \alpha_p^{2i(i+2)} (1 - (-1)^{j-i} \alpha_p^{2j^2+4j-2i^2-4i}) \\ &\sim 1 - (-1)^{j-i} \alpha_p^{2(j-i)(i+j+2)}. \end{aligned}$$

Now we have three cases:

1. The hypothesis implies,

$$\mu_i - \mu_j \sim 1 - (-\alpha_p^{2(j-i)(i+j+2)})$$

is a unit by Proposition (2.3.1), as $-\alpha_p^{2(j-i)(i+j+2)}$ has order divisible by two distinct primes.

2. The hypothesis implies,

$$\mu_i - \mu_j \sim 1 \pm i \sim \sqrt{2}.$$

3. Finally the hypothesis implies that for some $k \leq r-1$,

$$\mu_i - \mu_j \sim 1 - \alpha_p^{8k} = (1 - \alpha_p^8)(1 + \alpha_p^8 + \dots + \alpha_p^{8(k-1)}) \sim (1 - \alpha_p^4)(1 + \alpha_p^4) \sim (1 + \alpha_p^4).$$

As $(1 + \alpha_p^8 + \dots + \alpha_p^{8(k-1)})$, and $(1 - \alpha_p^4)$ are units by [19, Lem 3.1(ii)], and Proposition (2.3.1) respectively.

■

Proposition 2.4.1 *The number of pairs (i, j) with $0 \leq i < j \leq r-2$ such that,*

$$1. \mu_i - \mu_j \sim \sqrt{2} \text{ is } (\frac{r-1}{2}).$$

$$2. \mu_i - \mu_j \sim 1 + \alpha_p^4 \text{ is } (\frac{r-1}{2})(\frac{r-3}{2}).$$

Proof. To prove the first part, we look at all pairs (i, j) with $(j-i)(i+j+2) \equiv 0 \pmod{r}$ which automatically will satisfy $i \not\equiv j \pmod{2}$. This implies that $i+j+2 = r$. So we have $(\frac{r-1}{2})$ -pairs of such (i, j) . Hence the first part follows. Now for every $0 \leq i \leq r-4$, there are $(\lceil \frac{r-3-i}{2} \rceil)$ - j 's such that $i \equiv j \pmod{2}$. Hence, we have $2(1+2+\dots+\frac{r-3}{2})$ -pairs of such (i, j) . Hence the second part follows. ■

Theorem 2.4.1 *Let $\mathcal{B}_p = \{t^i(\omega_p) \mid 0 \leq i \leq d_p-1\}$. Then \mathcal{B}_p is a basis for $\mathcal{S}_p(S^1 \times S^1)$.*

Proof. We have

$$t^j(\omega_p) = \mathcal{D}_p^{-1} \sum_{i=0}^{d_p-1} (-1)^i [i+1] \mu_i^j e_i.$$

Let W be the matrix which expresses \mathcal{B}_p in terms of $\{e_0, e_1, \dots, e_{d_p-1}\}$. The determinant of W is a unit (product of $(-1)^i [i+1]$, see Corollary (2.7.1)) times $\mathcal{D}_p^{-d_p}$ times

the determinant of the Vandermonde matrix $[\mu_i^j]$, where $0 \leq i, j \leq d_p - 1 = r - 2$. By the previous lemma

$$\det[\mu_i^j] = \pm \prod_{i < j} (\mu_i - \mu_j) \sim \sqrt{2}^{\binom{r-1}{2}} (1 + \alpha_p^4)^{\binom{r-1}{2} \binom{r-3}{2}}.$$

As $\mathcal{D}_p \sim \sqrt{2}(1 + \alpha_p^4)^{\binom{r-3}{2}}$, we conclude

$$\det W \sim (\sqrt{2})^{-\binom{r-1}{2}} (1 + \alpha_p^4)^{-\binom{r-1}{2} \binom{r-3}{2}}. \quad (2.5)$$

As the determinant of W is non-zero, we conclude that \mathcal{B}_p is linearly independent. Now by Proposition (2.3.2), we know $(e_i, e_i) \sim \mathcal{D}_p$. Therefore the determinant of the form (2.2) with respect to the set $\{e_0, e_1, \dots, e_{d_p-1}\}$ is $\mathcal{D}_p^{r-1} \sim (\sqrt{2}(1 + \alpha_p^4)^{\binom{r-3}{2}})^{(r-1)}$. By equation (2.5) and the fact $1 + \alpha_p^4 \sim \overline{1 + \alpha_p^4} = 1 + \alpha_p^{-4}$, the determinant of the form (2.2) with respect to \mathcal{B}_p is a unit. Let \mathcal{W} denote the \mathcal{O}_p -submodule of $\mathcal{S}_p(S^1 \times S^1)$ generated by \mathcal{B}_p . We can conclude that the form on \mathcal{W} is unimodular. Hence $\mathcal{W} = \mathcal{W}^\sharp$ and so the set \mathcal{B}_p forms a basis for $\mathcal{S}_p(S^1 \times S^1)$ by equation (2.3). ■

Remark 2.4.1 *This theorem and its proof are analogous to [13, Thm. 6.1] and its proof.*

Corollary 2.4.1 *$\mathcal{S}_p(\Sigma)$ is generated by 3-manifolds (with no banded links) with boundary Σ .*

Proof. We expand the graph in every element in the Proposition (2.3.3) in terms of linear combinations of banded links (with some simple curves are colored ω_p). Then we replace any link component (that is not colored ω_p) by a linear combination from the set $\{t^i(\omega_p) \mid 0 \leq i \leq d_p - 1\}$. Hence the result follows by doing the required surgery on all the components of the link in every summand. ■

Remark 2.4.2 *The above result is true if we replace p by an odd prime as a corollary of [13, Thm. 6.1].*

2.5 The Frohman Kania-Bartoszyńska Ideal

We can apply the results from the previous section to compute the Frohman Kania-Bartoszyńska ideal using the $SU(2)$ -theory for a special family of 3-manifolds with torus boundary. Before we do so, we review this ideal.

Definition 2.5.1 ([7]) *Let N be a 3-manifold with boundary, we define $\mathcal{J}_p(N)$ to be the ideal generated over \mathcal{O}_p by*

$$\{I_p(M) \mid \text{where } M \text{ is a closed connected 3-manifold containing } N\}.$$

The importance of this ideal is in being an invariant of 3-manifolds (with boundary) and an obstruction to embedding as stated in the following propositions.

Proposition 2.5.1 ([7]) *The ideal \mathcal{J}_p is an invariant of oriented 3-manifolds with boundary.*

Proposition 2.5.2 ([7]) *If N_1, N_2 are an oriented compact 3-manifolds, and N_1 embeds in N_2 , then $\mathcal{J}_p(N_2) \subset \mathcal{J}_p(N_1)$.*

Remark 2.5.1 *Frohman and Kania-Bartoszyńska defined this ideal using the $SU(2)$ -TQFT-theory. Afterwards, Gilmer defined this ideal using the $SO(3)$ -TQFT-theory and the 2-theory.*

In general, it is not easy to compute this ideal because we have infinitely many closed connected 3-manifolds that contains N . Following his work with Masbaum in the case p an odd prime, Gilmer observed that $\mathcal{J}_p(N)$ is finitely generated based on his result that $\mathcal{S}_p(\Sigma)$ is finitely generated in the case p twice an odd prime as well. We give a finite set of generators for this ideal for any oriented compact 3-manifold using the $SU(2)$ -TQFT-theory which can be obtained by the following construction.

Definition 2.5.2 *Assume L is an ordered link of two components K, J . Let N_L be the manifold obtained by doing surgery in S^3 along K in the complement of J .*

Proposition 2.5.3

$$\mathcal{J}_p(N_L) = \langle I_p(M_i) \mid 0 \leq i \leq d_p - 1 \rangle,$$

where M_i is the 3-manifold obtained by doing surgery along the component K and the component J with framing i in S^3 .

Proof. If p is an odd prime this was proved in [12]. With the help of Theorem (2.4.1), the case p twice an odd prime follows in the same way. ■

2.6 The Quantization Functors for $p = 1$, and 2

In order to understand the relation between \mathcal{J}_r and \mathcal{J}_{2r} when r is an odd prime. We consider the theories associated to $p = 1$ and $p = 2$.

We begin by reviewing the quantization functor for $p = 1$ in detail. We start by listing the ring $k_1 = \mathbb{Z}$, and the surgery element $\Omega_1 = \omega_1 = 1$ for this theory defined in [1]. We also have $\kappa_1 = \mathcal{D}_1 = \theta_1 = 1$. One has $I_1(M) = \langle M \rangle_1 = \theta_1(M) = (-2)^{\#k}$ where $\#k$ is the number of components of the banded link in a closed 3-manifold M . Then (by [3, Prop. 1.1]) there exists a unique cobordism generated quantization functor (V_1, Z_1) that extends this invariant. In fact, this quantization functor can be described explicitly for surfaces as follows. $V_1(\Sigma)$ is the quotient of the \mathbb{Z} -module generated by all 3-manifolds (with banded links) with boundary Σ by the radical of the following form

$$\langle \ , \ \rangle_\Sigma : V_1(\Sigma) \times V_1(\Sigma) \rightarrow \mathbb{Z},$$

given by

$$\langle [M_1], [M_2] \rangle_\Sigma = \langle M_1 \cup_\Sigma -M_2 \rangle_1.$$

This module is isomorphic to \mathbb{Z} with any handlebody whose boundary is Σ as a generator. Hence if $M : \Sigma \rightarrow \Sigma'$, then $Z_1(M) : V_1(\Sigma) \rightarrow V_1(\Sigma')$ is the just the multiplication by $(-2)^{\#k}$.

Now we consider the quantization functor for $p = 2$. We start by introducing the ring and its ring of integers used in this theory

$$k_2 = \mathbb{Z}[\alpha_2, \frac{1}{2}] \quad \text{and} \quad \mathcal{O}_2 = \mathbb{Z}[\alpha_2].$$

The surgery element for this theory is $\omega_2 = \frac{1}{\sqrt{2}}\Omega_2$ where $\Omega_2 = 1 + \frac{\sqrt{2}}{2}$ defined in [1]. One has $\mathcal{D}_2 = \sqrt{2}$, and $\kappa_2 = \zeta_8$. Therefore the invariant of a closed connected

3-manifold M , which is obtained by doing surgery on S^3 along the link L , in terms of ω_2 is given by

$$\langle M \rangle_2 = \frac{1}{\sqrt{2}} \kappa_2^{\sigma(L)} \langle L(\omega_2) \rangle, \text{ where } \langle \rangle \text{ denotes the Kauffman bracket.} \quad (2.6)$$

From this formula, we can easily verify that

$$\langle M_1 \sharp M_2 \rangle_2 = \sqrt{2} \langle M_1 \rangle_2 \langle M_2 \rangle_2. \quad (2.7)$$

Now this invariant $\langle M \rangle_2$ defined in [3, §2] is involutive and extended to be multiplicative, hence (by [3, Prop. 1.1]) there exists a unique cobordism generated quantization functor that extends $\langle M \rangle_2$ which is denoted by (V_2, Z_2) . The modules $V_2(\Sigma)$ carry a Hermitian sesquilinear form defined as follows.

$$\langle \cdot, \cdot \rangle_\Sigma : V_2(\Sigma) \times V_2(\Sigma) \rightarrow k_2,$$

given by

$$\langle [M_1], [M_2] \rangle_\Sigma = \langle M_1 \cup_\Sigma -M_2 \rangle_2.$$

By [3, 1.5 and 6.3], $V_2(S^1 \times S^1)$ is generated by two elements each of which is a solid torus where the core is colored either 0 or 1. The pairing in terms of this basis is given by

$$\begin{aligned} \langle 1, 1 \rangle_{S^1 \times S^1} &= \langle S^1 \times S^2 \rangle_2 = 1. \\ \langle 1, z \rangle_{S^1 \times S^1} &= \langle (S^1 \times S^2, z) \rangle_2 = \frac{1}{\sqrt{2}} \langle H \rangle = \frac{1}{2} [2 + (-2)] = 0 = \overline{\langle z, 1 \rangle}_{S^1 \times S^1}. \end{aligned}$$

Here H is the Hopf link with one of the components is colored ω_2 . Finally,

$$\langle z, z \rangle_{S^1 \times S^1} = \langle (S^1 \times S^2, z \sqcup z) \rangle_2 = \frac{1}{\sqrt{2}} \langle K \rangle = \frac{1}{2} [4 + 4] = 4.$$

Here K is the 3-chain link where the middle chain is colored ω_2 .

Hence the matrix of the form $\langle \cdot, \cdot \rangle_{S^1 \times S^1}$ in terms of this basis is given by $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

If we restrict this theory to the category of nonempty connected objects and connected cobordisms between them, then we have an integral cobordism theory as before.

This follows from the fact $\langle \cdot \rangle_2$ is integral as stated in the proof of [19, Thm. 1.1].

Definition 2.6.1 *We define $\mathcal{S}_2(\Sigma)$ to be the \mathcal{O}_2 -submodule of $V_2(\Sigma)$ generated by all connected vacuum states, and we define an \mathcal{O}_2 -Hermitian sesquilinear form on $\mathcal{S}_2(\Sigma)$ given by $(\cdot, \cdot)_\Sigma = \sqrt{2} \langle \cdot, \cdot \rangle_\Sigma$.*

Remark 2.6.1 One could similarly define $\mathcal{S}'_2(\Sigma)$ based on the invariant $\langle \rangle'_2$ defined in [3, §. 1.B]. In this case, the basis $\{1, z\}$ for $V'_2(S^1 \times S^1)$ over k_2 is also a basis for $\mathcal{S}'_2(S^1 \times S^1)$ over \mathcal{O}_2 . However, this theory is not useful for us in this chapter.

The above basis for $V_2(S^1 \times S^1)$ does not generate $\mathcal{S}_2(S^1 \times S^1)$. The following theorem gives a basis.

Theorem 2.6.1 Assume that t is the twist map defined in [1], and $\mathcal{B} = \{\omega_2, t(\omega_2)\}$. Then \mathcal{B} is a basis for $\mathcal{S}_2(S^1 \times S^1)$, and the form is unimodular on $\mathcal{S}_2(S^1 \times S^1)$. Moreover, the matrix of the form defined in the previous definition in terms of \mathcal{B} is given by
$$\begin{pmatrix} \sqrt{2} & \frac{1-i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & \sqrt{2} \end{pmatrix}.$$

Proof. Let ω_2 and $t(\omega_2)$ stands for the elements in the Kauffman skein module of the solid torus where the core is colored ω_2 and $t(\omega_2)$ respectively. From the definition we know that these two elements lie in $\mathcal{S}_2(S^1 \times S^1)$, hence $\mathcal{W} = \text{Span}_{\mathcal{O}_2} \mathcal{B} \subseteq \mathcal{S}_2(S^1 \times S^1)$. The matrix of the form $(\ , \)_{S^1 \times S^1}$ is given by $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 4\sqrt{2} \end{pmatrix}$, and since the matrix of \mathcal{B} in terms of $\{1, z\}$ is given by $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{2\sqrt{2}} \end{pmatrix}$. Then the matrix B of the form in terms of \mathcal{B} is given by

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 4\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{2\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \frac{1-i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & \sqrt{2} \end{pmatrix}.$$

So the form restricted on \mathcal{W} has a unit determinant. Hence $\mathcal{W} = \mathcal{W}^\#$. Using equation (2.3), we get that \mathcal{W} is all of $\mathcal{S}_2(S^1 \times S^1)$. In conclusion, $\{\omega_2, t(\omega_2)\}$ is a basis for $\mathcal{S}_2(S^1 \times S^1)$. ■

Definition 2.6.2 Let $H_{i_1 i_2 \dots i_g}$ be the boundary connected sum of g solid tori where the core of the m -th torus is colored $i_m = 0$ or 1 . Also, let

$$\mathcal{B} = \{H_{i_1 i_2 \dots i_g} \mid (i_1, i_2, \dots, i_g) \text{ is a } g\text{-tuple over } \{0, 1\}\}.$$

This set \mathcal{B} is an orthogonal basis for $V_2(\Sigma)$, and the pairing is described as follows:

Proposition 2.6.1 The above set \mathcal{B} forms an orthogonal basis with respect to the form $\langle \ , \ \rangle_2$ given by

$$\langle H_{i_1 i_2 \dots i_g}, H_{j_1 j_2 \dots j_g} \rangle_\Sigma = 4^k (\sqrt{2})^{g-1},$$

where $k = i_1 + i_2 + \dots + i_g$.

Proof. By [3, 1.5, and 6.3] \mathcal{B} is a basis. The result now follows from equation (2.7), and the computations for $V_2(S^1 \times S^1)$ after that equation. ■

We can describe $\mathcal{S}_2(\Sigma)$ as the \mathcal{O}_2 -submodule of $V_2(\Sigma)$ generated by all 3-manifolds with boundary Σ and links sitting inside of them. As $z = 2\sqrt{2}\omega_2 - 2$, one has a similar result to Corollary (2.4.1) for this theory.

Definition 2.6.3 Let $H'_{i_1 i_2 \dots i_g}$ be the boundary connected sum of g solid tori where the core of the m -th torus is colored $t^{i_m}(\omega_2)$ for $i_m = 0$, or 1.

Also, let

$$\mathcal{B}' = \{H'_{i_1 i_2 \dots i_g} \mid (i_1, i_2, \dots, i_g) \text{ is a } g\text{-tuple over } \{0, 1\}\}.$$

Theorem 2.6.2 The above set \mathcal{B}' forms a basis for $\mathcal{S}_2(\Sigma)$.

Proof. Let $(S^1 \times S^2)_{ij}$ denote $S^1 \times S^2$ formed by gluing two solid tori whose cores are colored $t^i(\omega_2)$, and $t^j(\omega_2)$ where $i, j \in \{0, 1\}$. Let us look at the pairing

$$\begin{aligned} (H'_{i_1 i_2 \dots i_g}, H'_{j_1 j_2 \dots j_g})_\Sigma &= \sqrt{2} \langle H'_{i_1 i_2 \dots i_g}, H'_{j_1 j_2 \dots j_g} \rangle_\Sigma \\ &= \sqrt{2} \langle \#_{k=1}^g (S^1 \times S^2)_{i_k j_k} \rangle_2 \\ &= \sqrt{2}^g \prod_{k=1}^g \langle (S^1 \times S^2)_{i_k j_k} \rangle_2 \\ &= \prod_{k=1}^g ((S^1 \times S^2, t^{i_k}(w_2) \sqcup t^{j_k}(w_2))_2). \end{aligned}$$

With a natural order, the matrix of the form in terms of this set is given by $\bigotimes^g B$ (B is defined in the proof of the previous theorem). This implies that the determinant of this form is a unit. By a similar argument as in the proof of Theorem (2.6.1), the module generated by this set is all of $\mathcal{S}_2(\Sigma)$. ■

We define $I_2(M) = \sqrt{2} \langle M_b \rangle_2$ for a closed 3-manifold M where $\langle M \rangle_2$ as defined in Equation (2.6). Also we define the Frohman Kania-Bartoszyńska ideal \mathcal{J}_2 just as in the previous section. Now we can compute this ideal easily for all 3-manifolds using the 2-theory by making use of above results. For example, we confirm a result of Gilmer and prove it using our basis.

Proposition 2.6.2 [9, Prop. 15] *Let N_L, L as defined in Definition (2.5.2). Also, let l be the linking number between K and J , and k is the framing of K . If l is odd then $\mathcal{J}_2(N_L) = \mathcal{O}_2$. If l is even, then we have the following:*

$$\mathcal{J}_2(N_L) = \begin{cases} (\sqrt{2}), & \text{if } k \equiv 0 \pmod{4}; \\ 0, & \text{if } k \equiv 2 \pmod{4}; \\ \mathcal{O}_2, & \text{if } k \equiv 1 \text{ or } 3 \pmod{4}. \end{cases}$$

Proof. From Theorem (2.6.1), we know that $\mathcal{J}_2(N_L)$ is generated by two elements. In fact, it is generated by $I_2(M_i)$ where M_i is the 3-manifold obtained by doing surgery along the component K and the component J with framing 0 or 1 in S^3 . Now, we use the formula in [2, Cor. 2.4] to compute the two generators of this ideal. The computations shows that (note that κ_2 is a unit in \mathcal{O}_2)

$$\mathcal{J}_2(N_L) = \langle \frac{1}{2}(i^{k+2l} + i^k + 2), \frac{1}{2}(i^{k+2l+2} + i + i^k + 1) \rangle,$$

Now if we consider all the possibilities, we obtain the required result. ■

Also, we compute this ideal for all 3-manifolds N_K that are obtained by doing surgery on a knot K in the complement of a tubular neighborhood of an eyeglass graph: $0 - 0$ in S^3 .

Proposition 2.6.3 *Let l_1 and l_2 be the linking numbers of K with the first and the second loops in the eyeglass respectively, and k is the framing of K . Then we have*

$$\mathcal{J}_2(N_K) = \begin{cases} (\sqrt{2}), & \text{if } l_1 \equiv l_2 \equiv 0 \pmod{2} \text{ and } k \equiv 0 \pmod{4}; \\ 0, & \text{if } l_1 \equiv l_2 \equiv 0 \pmod{2} \text{ and } k \equiv 2 \pmod{4}; \\ \mathcal{O}_2, & \text{if any of } l_1, l_2, k \text{ is odd.} \end{cases}$$

Proof. Let m be the linking number between the loops. From Theorem (2.6.2), we know $\mathcal{J}_2(N_K)$ is generated by four elements. In fact, it is generated by $I_2(M_{i,j})$ where $M_{i,j}$ is the 3-manifold obtained by doing surgery along the component K and the loops

with framing $i, j = 0$ or 1 in S^3 . As in the proof of the previous proposition, one sees

$$\begin{aligned} \mathcal{J}_2(N_K) = & \left\langle \frac{1}{2\sqrt{2}}(3 + i^k + i^{2m} + i^{k+2l_1} + i^{k+2l_2} + i^{k+2l_1+2l_2+2m}), \right. \\ & \frac{1}{2\sqrt{2}}(2 + i + i^k + i^{2m+1} + i^{k+2l_1} + i^{k+2l_2+1} + i^{k+2l_1+2l_2+2m+1}), \\ & \frac{1}{2\sqrt{2}}(2 + i + i^k + i^{2m+1} + i^{k+2l_1+1} + i^{k+2l_2} + i^{k+2l_1+2l_2+2m+1}), \\ & \left. \frac{1}{2\sqrt{2}}(1 + 2i + i^k + i^{2m+2} + i^{k+2l_1+1} + i^{k+2l_2+1} + i^{k+2l_1+2l_2+2m+2}) \right\rangle. \end{aligned}$$

If we take all possibilities, we get the required result. ■

2.7 Relating the r -th and $2r$ -th Theories

For the rest of this chapter, we assume that r is an odd prime and $p = 2r$.

Remark 2.7.1 *The results of this section are slight variations of results of [3, §. 6] and [2, §. 2]. The ring k_p is not exactly the same as the ring denoted this way in [3].*

The ring k_p will be considered as a k_2 (or a k_r)-module via the homomorphisms defined below. The following is a slight variation of the maps defined in [2, §. 2].

Lemma 2.7.1 *There are well-defined ring homomorphisms $i_r : k_2 \rightarrow k_p$, $j_r : k_r \rightarrow k_p$ given by*

$$\begin{aligned} i_r(\alpha_2) &= \alpha_p^{r^2}, \quad j_r(\alpha_r) = \alpha_p^{1+r^2} \text{ for } r \equiv 1 \pmod{4}, \text{ and} \\ j_r(A_r) &= A_p^{1+r^2} \text{ for } r \equiv -1 \pmod{4}. \end{aligned}$$

We need the following remark to prove that these maps are well-defined.

Remark 2.7.2 *If α is a primitive n -th root of unity, then α^m is a primitive $\frac{n}{\gcd(n,m)}$ -th root of unity.*

Proof. To prove that the map i_r is a well-defined ring homomorphism, we show $\alpha_p^{r^2}$ is a primitive $8r$ -th root of unity. This is true, as $\gcd(8r, r^2) = r$ and α_p is a primitive $8r$ -th root of unity. Similarly for j_r but we consider two cases:

1. For $r \equiv 1 \pmod{4}$, we have $\alpha_p^{1+r^2}$ is a primitive $4r$ -th root of unity, as $\gcd(8r, 1+r^2) = 2$ and α_p is a primitive $8r$ -th root of unity.

2. For $r \equiv -1 \pmod{4}$, we have $A_p^{1+r^2}$ is a primitive $2r$ -th root of unity, as $\gcd(4r, 1+r^2) = 2$ and A_p is a primitive $4r$ -th root of unity.

■

Corollary 2.7.1 *The quantum integers $[i]_p, 1 \leq i \leq d_p$ are units in \mathcal{O}_p .*

Proof. We know that the quantum integers $[i]_r$ for $1 \leq i \leq r-1$ are units in the \mathcal{O}_r see [13, Lem. 4.1(iii)] and [19, Lem. 3.1(ii)]. So we conclude that $[i]_p$ are units for $1 \leq i \leq r-1 = d_p$, as

$$j_r([i]_r) = (-1)^i [i]_p, \quad \forall 1 \leq i \leq d_p,$$

since

$$j_r(A_r^2) = -A_p^2.$$

■

Given any k_2 (or k_r)-module, we can define a k_p -module by tensoring the original module with k_p over k_2 (or k_r) respectively. We let $\widehat{V}_2(\Sigma)$ (or $\widehat{V}_r(\Sigma)$) be the k_p -module obtained in this way. We give a relation between $V_1, \widehat{V}_2, \widehat{V}_r$, and V_{2r} for any surface Σ , but before that we need the following slight reformulation of [2, Thm. 2.1].

Theorem 2.7.1 *For any closed 3-manifold M with possibly a banded link sitting inside of it we have,*

$$I_1(M)I_{2r}(M) = i_r(I_2(M))j_r(I_r(M)). \quad (2.8)$$

Proof. Theorem (2.1) in [2] states the following:

$$\theta_1(M)\theta_{2r}(M) = i_r(\theta_2(M))j_r(\theta_r(M)). \quad (2.9)$$

Letting $M = S^3$, we get

$$\mathcal{D}_{2r}^{-1} = i_r(\mathcal{D}_2^{-1})j_r(\mathcal{D}_r^{-1}),$$

as $\theta_1(S^3) = 1$. Now multiply both sides of equation (2.9) by $\mathcal{D}_{2r}^{-\beta_1(M)}$, and replace it by $i_r(\mathcal{D}_2^{-\beta_1(M)})j_r(\mathcal{D}_r^{-\beta_1(M)})$ in the right hand side. Then the result follows from the relation between θ and I . ■

We let κ_n to be an element that plays the role of κ^3 in [3]. We define this element as follows:

$$\kappa_n = \begin{cases} \alpha_{4n}^{-6 - \frac{n(n+1)}{2}}, & \text{if } n \text{ is an odd prime;} \\ -\alpha_{4n}^{-6 - \frac{n(n+1)}{2}}, & \text{if } n \text{ is twice an odd prime.} \end{cases}$$

Changing the weight by one multiplies the invariant $\langle \rangle_n$ by κ_n .

Lemma 2.7.2 *For the above ring homomorphisms. We have*

$$\kappa_p = i_r(\kappa_2)j_r(\kappa_r)$$

Proof. We have

$$i_r(\kappa_2)j_r(\kappa_r) = i_r(\alpha_2)j_r(\alpha_r^{-6 - \frac{r(r+1)}{2}}) = \alpha_p^{r^2}(\alpha_p^{-6 - \frac{r(r+1)}{2}})^{1+r^2} = -\alpha_p^{-6 - \frac{2r(2r+1)}{2}} = \kappa_p,$$

as

$$r^2 + (-6 - \frac{r(r+1)}{2})(1+r^2) \equiv -6 - \frac{2r(2r+1)}{2} + 4r \pmod{8r}$$

■

We are now able to give the proof of a result used in §2.2.

Proof of Lemma 2.3.1. The first case follows from [13, Lem. 4.1(ii)]. The second case follows from the following facts from the proof of Corollary (2.7.1) and from Theorem (2.7.1).

1. $\mathcal{D}_p = \mathcal{D}_{2r} = i_r(\mathcal{D}_2)j_r(\mathcal{D}_r)$, where $\mathcal{D}_2 = \sqrt{2}$.
2. $j_r(A_r^2) = -A_{2r}^2 = -\alpha_{2r}^4$.

■

Theorem 2.7.2 *There is a natural k_p -isomorphism $F : V_1(\Sigma) \otimes V_p(\Sigma) \rightarrow \widehat{V}_2(\Sigma) \otimes \widehat{V}_r(\Sigma)$ such that*

$$F([M]_1 \otimes [M]_p) = [M]_2 \otimes [M]_r, \tag{2.10}$$

where M is a 3-manifold with banded link (but not linear combination of links) sitting inside of it.

Corollary 2.7.2 *The map in the previous theorem defines a k_p -isomorphism between $V_p(\Sigma)$, and $\widehat{V}_2(\Sigma) \otimes \widehat{V}_r(\Sigma)$.*

To prove this theorem, we use the following version of [3, Lemm. 6.4].

Lemma 2.7.3 *Let \mathcal{V}, W be free modules over an integral domain R (with involution) equipped with Hermitian sesquilinear forms $\langle \cdot, \cdot \rangle_{\mathcal{V}}, \langle \cdot, \cdot \rangle_W$, and let $F : \mathcal{V} \rightarrow W$ be a form-preserving linear map. Let $(V, \langle \cdot, \cdot \rangle_V)$ be the quotient of \mathcal{V} by the radical of $\langle \cdot, \cdot \rangle_{\mathcal{V}}$. Suppose that $\langle \cdot, \cdot \rangle_W$ is non-degenerate. Suppose either that F is surjective, or V and W are free of finite rank and $\langle \cdot, \cdot \rangle_V$ is unimodular and furthermore that $\text{rank}(W) \leq \text{rank}(V)$. Then F induces an isometry $F_{ind} : V \rightarrow W$.*

Proof. We use f to denote the map between $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ induced by F . Let us assume that F is surjective. To show that F induces an isometry, i.e f is an isometry, we show that f is injective. Let $x \in \ker(f)$, then $\langle f(x), w \rangle_W = 0, \forall w \in W$. Since f is form preserving, this implies that $\langle x, v \rangle_V = 0 \forall v \in V$, i.e x in the radical of $\langle \cdot, \cdot \rangle_V$. So $x = 0$, as the form on V is non-degenerate. Hence f is an isometry.

Now when V and W are free of finite rank and $\langle \cdot, \cdot \rangle_V$ is unimodular and furthermore that $\text{rank}(W) \leq \text{rank}(V)$. Let x , and y be two elements in V with $f(x) = f(y)$. As f is form preserving, we conclude that $x^* = y^*$. Here and elsewhere, v^* is a map from $V \rightarrow R$ defined by $v^*(z) = \langle z, v \rangle_V$ for any $v, z \in V$. This implies that $x = y$, since the form $\langle \cdot, \cdot \rangle_V$ is unimodular (in fact non-degenerate is enough). Hence f is injective. Now let $w \in W$. Since the form on V is unimodular $\exists v \in V \ni v^* = f^*(w^*)$. We claim that $f(v) = w$. To prove this claim, we know that $f^*(f(v)^*) = v^*$ as

$$f^*(f(v)^*)(z) = f(v)^*(f(z)) = \langle f(z), f(v) \rangle_W = \langle z, v \rangle_V = v^*(z) \text{ for } z \in V.$$

Let K be the field of fraction of R , then $f \otimes id_K : V \otimes K \rightarrow W \otimes K$ is injective, as f is injective. As $\dim(V \otimes K) \geq \dim(W \otimes K)$, $f \otimes id_K$ is an isomorphism. Thus $(f \otimes id_K)^*$ and thus f^* is injective. This implies that $f(v)^* = w^*$, as $f^*(f(v)^*) = v^* = f^*(w^*)$. So we conclude that $f(v) = w$, i.e f is surjective. Hence f is an isometry. ■

Proof of Theorem 2.7.2. It follows from Theorem (2.7.1) and Lemma (2.7.2) that formula (2.10) defines a form-preserving linear map. We know already that the form on $\widehat{V}_2(\Sigma) \otimes \widehat{V}_r(\Sigma)$ is non-degenerate. Finally, we have two cases namely,

- If $r = 1$, then F is just flipping the tensors. Hence, it is an isometry

- If $r \geq 3$, then the result follows from the fact that $\text{rank}(V_1(\Sigma) \otimes V_{2r}(\Sigma)) = \text{rank}(\widehat{V}_2(\Sigma) \otimes \widehat{V}_r(\Sigma))$, and the second part of the lemma.

■

2.8 The Second Basis for $\mathcal{S}_p(S^1 \times S^1)$

We give new basis for $\mathcal{S}_r(S^1 \times S^1)$ that will be used in constructing another basis for $\mathcal{S}_p(S^1 \times S^1)$. To do so, we need the following lemma.

Lemma 2.8.1 *If $0 \leq i < j \leq \frac{r-3}{2}$, then the twist coefficients satisfy $\mu_i^2 - \mu_j^2 \sim \mu_i - \mu_j \sim 1 - A_r^2$.*

Proof. Notice that $\mu_i = q_r^{i^2+2i}$ where q_r denotes the primitive r -th root of unity given by $-A_r$.

$$\begin{aligned} \mu_i^2 - \mu_j^2 &= (\mu_i + \mu_j)(\mu_i - \mu_j) = (q_r^{i^2+2i} + q_r^{j^2+2j})(\mu_i - \mu_j) \sim \\ &\quad (1 + q_r^{(j-i)(j+i+2)})(\mu_i - \mu_j) \sim \mu_i - \mu_j \sim 1 - A_r^2. \end{aligned}$$

We used the result of the fourth part of [13, Lem. (4.1)] in the last equality up to a unit. Also in the one next to last, we used the fact that $1 + q_r^{(j-i)(j+i+2)}$ is a unit by [19, Lem. (3.1)] as $\text{gcd}(r, (j-i)(j+i+2)) = 1$. ■

Theorem 2.8.1 *Let $\mathcal{B}_{2r} = \{t^{2i}(\omega_r) \mid 0 \leq i \leq d_r - 1\}$, and $\mathcal{B}_{2r+1} = \{t^{2i+1}(\omega_r) \mid 0 \leq i \leq d_r - 1\}$. Then \mathcal{B}_{2r} and \mathcal{B}_{2r+1} form bases for $\mathcal{S}_r(S^1 \times S^1)$.*

Proof. The proof of [13, Thm. 4.1] now goes through with μ_i^2 playing the role of μ_i and t^{2j} playing the role of t^j . We use the previous lemma when appropriate to obtain that \mathcal{B}_{2r} is a basis. To prove that \mathcal{B}_{2r+1} is a basis, we use the fact that the twist map t is an isomorphism of the Kauffman skein module of the solid torus. ■

Notation: We use the notation $\widehat{\mathcal{S}}_i(S^1 \times S^1) = \mathcal{S}_i(S^1 \times S^1) \otimes_{k_i} k_p$ for $i = 2$ or r .

Definition 2.8.1 *Let*

$$\delta_i = \begin{cases} 0, & \text{if } i + p \equiv 2 \text{ or } 3 \pmod{4}; \\ 1, & \text{if } i + p \equiv 0 \text{ or } 1 \pmod{4}. \end{cases}$$

We defined δ_i so that the following two lemmas hold.

Lemma 2.8.2 *If $i + \delta_i p \equiv 0 \pmod{2}$, then $i + \delta_i p \equiv 0 \pmod{4}$.*

Proof. We know that $p \equiv 2 \pmod{4}$, now we have two cases to consider

- If $\delta_i = 0$, then $i + p \equiv 2 \pmod{4}$ as i and p are even. So we conclude $i + \delta_i p = i \equiv 0 \pmod{4}$.
- If $\delta_i = 1$, then $i + p \equiv 0 \pmod{4}$ as $i + p$ is even. So we conclude $i + \delta_i p \equiv i + p \equiv 0 \pmod{4}$.

■

Lemma 2.8.3 *If $i + \delta_i p \equiv 1 \pmod{2}$, then $i + \delta_i p \equiv 1 \pmod{4}$.*

A similar proof can be given for this lemma. The following theorem gives another basis for $\mathcal{S}_p(S^1 \times S^1)$.

Theorem 2.8.2 *Let $\mathcal{B}_p = \{t^{i+\delta_i p}(\omega_p) \mid 0 \leq i \leq d_p - 1\}$. Then \mathcal{B}_p is a basis for $\mathcal{S}_p(S^1 \times S^1)$.*

Proof.

We have that $\text{Span}_{\mathcal{O}_p} \mathcal{B}_p \subseteq \mathcal{S}_p(S^1 \times S^1)$, and $F(1 \otimes \mathcal{S}_p(S^1 \times S^1)) \subseteq \widehat{\mathcal{S}}_2(S^1 \times S^1) \otimes \widehat{\mathcal{S}}_r(S^1 \times S^1)$ where F is the map defined in formula (2.10). It is enough now to show that $F(1 \otimes \mathcal{B}_p)$ generates $\widehat{\mathcal{S}}_2(S^1 \times S^1) \otimes \widehat{\mathcal{S}}_r(S^1 \times S^1)$ which implies that $F(1 \otimes \mathcal{S}_p(S^1 \times S^1)) \subseteq \text{Span}_{\mathcal{O}_p} F(1 \otimes \mathcal{B}_p)$, i.e $\mathcal{S}_p(S^1 \times S^1) \subseteq \text{Span}_{\mathcal{O}_p} \mathcal{B}_p$. Hence, we conclude that \mathcal{B}_p is a basis for $\mathcal{S}_p(S^1 \times S^1)$ as $\text{rank}(\mathcal{S}_p(S^1 \times S^1)) = d_p$. To prove the claim, let us look at the image of \mathcal{B}_p under F .

$$t^{i+\delta_i p}(\omega_p) \rightarrow t^{i+\delta_i p}(\omega_2) \otimes t^{i+\delta_i p}(\omega_r)$$

where $i \in \{0, 1, \dots, d_p - 1\}$.

Let us consider first all the elements of \mathcal{B}_p with even number of twists, i.e $i + \delta_i p \equiv 0 \pmod{2}$. By Lemma (2.8.2), we get that $i + \delta_i p \equiv 0 \pmod{4}$. Hence those elements get mapped to $t^{4m}(\omega_2) = \omega_2$ for some m , as t^4 is the identity map in the 2-theory. Also they get mapped to $t^{2j}(\omega_r)$ for some $0 \leq j \leq d_p - 1$, as t^p is the identity map in the $SO(3)$ -TQFT-theory and i is even. The later elements form the basis \mathcal{B}_{2r} defined in the previous theorem. In short, the above elements get mapped to $\omega_2 \otimes \mathcal{B}_{2r}$.

Now we consider the elements of \mathcal{B}_p with odd number of twists, i.e $i + \delta_i p \equiv 1 \pmod{2}$. By Lemma (2.8.3), we get that $i + \delta_i p \equiv 1 \pmod{4}$. Hence those elements

get mapped to $t^{4m+1}(\omega_2) = t(\omega_2)$ for some m , as t^4 is the identity map in the 2-theory. Also they get mapped to $t^{2j+1}(\omega_r)$ for some $0 \leq j \leq d_p - 1$, as t^p is the identity map in the $SO(3)$ -TQFT-theory and i is odd. The later elements form the basis \mathcal{B}_{2r+1} defined in the previous theorem. In short, the above elements get mapped to $\omega_2 \otimes \mathcal{B}_{2r+1}$.

Hence the image of \mathcal{B}_p under F is a basis for $\widehat{\mathcal{S}}_2(S^1 \times S^1) \otimes \widehat{\mathcal{S}}_r(S^1 \times S^1)$, i.e generates it as required. ■

Corollary 2.8.1 *From the above proof, we conclude $\mathcal{S}_p(S^1 \times S^1) \cong \widehat{\mathcal{S}}_2(S^1 \times S^1) \otimes \widehat{\mathcal{S}}_r(S^1 \times S^1)$.*

We do not know if this holds for higher genus surfaces, but it is clear that $\mathcal{S}_p(\Sigma)$ maps into $\widehat{\mathcal{S}}_2(\Sigma) \otimes \widehat{\mathcal{S}}_r(\Sigma)$ under the map F .

Proposition 2.8.1

$$\mathcal{J}_p(N_L) = \langle I_p(M_{i+\delta_{ip}}) \mid 0 \leq i \leq d_p - 1 \rangle,$$

where M_i is the 3-manifold obtained by doing surgery along the component K and the component J with framing $i + \delta_{ip}$ in S^3 .

Finally, a good question would be: “Is there a relation between the Frohman Kania-Bartoszyńska ideals in the $SU(2)$ - and the $SO(3)$ -TQFT-theories?” An answer is given by the following theorem.

Theorem 2.8.3 *Let N be an oriented compact 3-manifold with boundary. Then we have*

$$\mathcal{J}_p(N) \subseteq i_r(\mathcal{J}_2(N))j_r(\mathcal{J}_r(N)),$$

where i_r and j_r are defined as in the previous section. Moreover, we have equality if N has a torus boundary.

Proof. To prove the inclusion, we have $F(1 \otimes \mathcal{S}_p(\Sigma)) \subseteq \widehat{\mathcal{S}}_2(\Sigma) \otimes \widehat{\mathcal{S}}_r(\Sigma)$. So if $[M]_p \in \mathcal{S}_p(\Sigma)$, then $F(1 \otimes [M]_p) = [M]_2 \otimes [M]_r \in \widehat{\mathcal{S}}_2(\Sigma) \otimes \widehat{\mathcal{S}}_r(\Sigma)$. By Theorem (2.7.1) as $I_1(M) = 1$, we have

$$I_p(N \cup_\Sigma -M) = i_r(I_2(N \cup_\Sigma -M))j_r(I_r(N \cup_\Sigma -M)) \in i_p(\mathcal{J}_2(N))j_p(\mathcal{J}_r(N)).$$

So we can conclude that $\mathcal{J}_p(N) \subseteq i_p(\mathcal{J}_2(N))j_p(\mathcal{J}_r(N))$. Now we prove the equality in the case of a torus boundary. Let $M_{i+\delta_{ip}}$ be the solid torus where its core colored $t^{i+\delta_{ip}}(w)$. From the previous proposition, we have

$$\begin{aligned}\mathcal{J}_p(N) &= \langle I_p(N \cup_{S^1 \times S^1} M_{i+\delta_{ip}}) \mid 0 \leq i \leq d_p - 1 \rangle \\ &= \langle i_r(I_2(N \cup_{S^1 \times S^1} M_{i+\delta_{ip}}))j_r(I_r(N \cup_{S^1 \times S^1} M_{i+\delta_{ip}})) \mid 0 \leq i \leq d_p - 1 \rangle \\ &= i_r(\mathcal{J}_2(N))j_r(\mathcal{J}_r(N)).\end{aligned}$$

The last equality follows as $F(1 \otimes \mathcal{B}_p)$ is a basis for $\widehat{\mathcal{S}}_2(S^1 \times S^1) \otimes \widehat{\mathcal{S}}_r(S^1 \times S^1)$. ■

Chapter 3

Periodic 3-Manifolds and Modular Categories

3.1 Introduction

Let p be an odd prime, and let G be the finite cyclic group \mathbb{Z}_p . We assume that all 3-manifolds are compact and closed. The quantum invariant of a 3-manifold can be defined using any modular category. In [8], Gilmer was interested in studying the relation between the $SU(2)$ -invariants of p -periodic 3-manifolds and their quotient manifolds. He obtained a congruence relating these invariants. His result was obtained by using the trace formula of topological quantum field theory (see proposition (3.4.1)) and studying Gaussian sums. Chbili used the results about the Jones polynomial and the Kauffman multi-bracket of p -periodic links to obtain a similar result for rational homology 3-spheres for the $SO(3)$ -invariants in [5]. Also in [4], he gave similar results for the $SU(3)$ and the MOO-invariants. Moreover in [6], Chen and Le generalized the above results for rational homology spheres using any complex simple Lie algebra. We give similar results for all 3-manifolds using any modular category over an integrally closed ground ring. Our proof takes place completely in the context of modular categories. We use the surgery descriptions of p -periodic 3-manifold and its orbit manifold, obtained in [20, 21], to prove the result.

In §3.1, we give a brief exposition on how to calculate the quantum invariant for any 3-manifold from its surgery description. In §3.2, we discuss the \mathbb{Z}_p -actions on 3-manifolds and the relation between the link that describes a p -periodic 3-manifold and the link that describes its orbit manifold. Some formulas and results regarding

the value of colored ribbon graphs under the covariant functor F will be given in §3.3.

Finally in §3.4, we state and prove the main result.

3.2 Quantum Invariants of 3-Manifolds

Fix a strict modular category $(\mathcal{V}, \{v_i\}_{i \in I})$ with ground ring K and a rank $\mathcal{D} \in K$. The material of this section is due to Turaev [22, Ch. II].

3.2.1 Introduction

A result due to Lickorish and Wallace asserts that every closed oriented 3-manifold can be obtained by surgery on S^3 along a framed link.

3.2.2 The τ -Invariant of Closed 3-Manifolds

Let M be a closed oriented 3-manifold obtained by surgery on S^3 along a framed link L . The τ -invariant of (M, Ω) associated to $(\mathcal{V}, \mathcal{D})$ where Ω is a colored ribbon graph in M is given by

$$\tau(M, \Omega) = \Delta^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \{L, \Omega\}. \quad (3.1)$$

Here $\sigma(L)$ is the signature of the linking matrix of the link L , and m is the number of components of L , and $\Delta = \{U^-\}$ where U^- denotes the diagram for the unknot with a single double point and writhe -1.

We use the notation

$$\begin{aligned} \{L, \Omega\} &= \sum_{\lambda \in \text{col}(L)} \{L, \Omega\}_\lambda \\ &= \sum_{\lambda \in \text{col}(L)} \prod_{i=1}^m \dim(\lambda(L_i)) F(\Gamma(L, \lambda) \cup \Omega), \end{aligned}$$

where $\text{col}(L)$ is the set of all mappings from the set of components of L to I (the set of simple objects), and $\Gamma(L, \lambda)$ is the ribbon graph obtained by coloring the i -th component of L by $V_{\lambda(i)}$. Here F is the covariant functor defined in [22, Ch. I] which assigns to a \mathcal{V} -colored ribbon graph in \mathbb{R}^3 an element of the ground ring.

3.2.3 The I -Invariant

We take the definition of the quantum invariant to be as follows

$$I(M, \Omega) = \mathcal{D}\tau(M, \Omega).$$

Our result is simpler when expressed using this normalization.

Theorem 3.2.1 [22, Th. 2.3.2] $\tau(M, \Omega)$ (or $I(M, \Omega)$) is a topological invariant of the pair (M, Ω) .

Example. We know that S^3 is obtained by doing surgery on the empty link, i.e $I(S^3) = 1$. Also, S^3 is obtained by doing surgery along the Hopf link H with framing 0 on both components. Hence, we conclude $\{H\} = \mathcal{D}^2$.

Corollary 3.2.1 $\{L, \Omega\}$ is invariant under Kirby sliding.

This corollary is really major part of proof of Theorem (3.2.1).

Finally, the τ -invariant can be recovered in terms of the TQFT-theory (V, Z) which is a functor from the category \mathcal{C} whose objects are closed surfaces and 3-manifolds as its morphisms (the surfaces and 3-manifolds have banded links sitting inside of them) to the category of K -modules and K -linear homomorphisms, where V is the functor on the surfaces and Z is the functor on 3-manifolds. In fact, the assigned value of a closed 3-manifold under Z is a scalar multiplication homomorphism from the base ring to itself and that scalar is the τ -invariant of that manifold. For more details of TQFT see [22, 3].

3.3 Periodic Links and Periodic 3-Manifolds

Let M be a closed oriented 3-manifold that is a result of surgery on S^3 along the framed link L .

Definition 3.3.1 A framed link L in S^3 is said to be p -periodic if there exists a \mathbb{Z}_p -action on S^3 , with a fixed point set equal to a circle, that maps L to itself under this action and L is assumed to be disjoint from the circle.

Definition 3.3.2 M is said to be p -periodic if there is an orientation preserving \mathbb{Z}_p -action with fixed point set equal to a circle, and the action is free outside this circle.

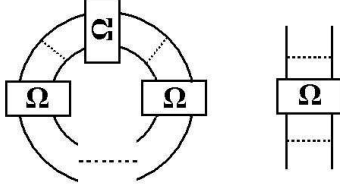


Figure 3.1: Periodic Link and its Quotient

Now we list the following two results from [20, 21] that will be used in later sections.

Theorem 3.3.1 *There is a \mathbb{Z}_p -action on M with a fixed point set equal to a circle iff M can be obtained as a result of surgery on a p -periodic link L in S^3 and \mathbb{Z}_p acts freely on the set of the components of L .*

By the positive solution of the Smith conjecture, we can represent any framed p -periodic link as a closure of some graph such that the rotation of this graph about the z -axis in \mathbb{R}^3 (or the circle in S^3) by $2\pi/p$ leaves it invariant, i.e $L_* = \overline{\Omega}$ (where the bar means the closure of the graph) see figure 3.1.

Let $M_* = M/\mathbb{Z}_p$ denote the orbit space, then M_* is obtained by surgery on S^3 along the link $L_* = L/\mathbb{Z}_p$.

Lemma 3.3.1 *Let L a p -periodic link in S^3 . The following are equivalent*

1. \mathbb{Z}_p acts freely on the set of components of L ;
2. the linking number of each component of the L_* the axis of the action is congruent to zero modulo p ;
3. the number of components of L is equal to p times the number of components of L_* .

3.4 Some Results About Traces

We use two different notions of trace. One is the trace of a linear homomorphism (denoted by Trace) in the category of K -modules and the other one is the trace of a

ribbon graph (denoted by Tr) in the category of ribbon graphs defined in [22, I.1.5]

Proposition 3.4.1

$$\tau(S^2 \times S^1, \overline{\Omega}) = \text{Trace}_{V(S^2, l)}(Z_{(S^2 \times I, \Omega)}),$$

where Ω is a colored ribbon $l \times l$ tangle in $S^2 \times I$.

Proof. This is a special case of the Trace Formula for TQFT [3, Prop. 1.2] and [22, Ex. 2.8.1]. ■

Lemma 3.4.1

$$\text{Tr}(\Omega) = \frac{1}{\mathcal{D}^2} \sum_{i \in I} \dim(V_i) \text{Trace}_{V(S^2, l+1)}(Z_{(S^2 \times I, 1_{V_i} \otimes \Omega)}).$$

Proof. Let H stands for the zero-framed Hopf link on both components. We have

$$\begin{aligned} \text{Tr}(F(\Omega)) &= F(\overline{\Omega}) \quad \text{by [22, Cor. 2.7.2]} \\ &= \frac{1}{\{H\}} \sum_{\lambda \in \text{col}(H)} \{H\}_\lambda F(\overline{\Omega}) , \quad \text{as } \{H\} = \sum_{\lambda \in \text{col}(L)} \{H\} \\ &= \frac{1}{\mathcal{D}^2} \sum_{\lambda \in \text{col}(H)} \dim(\lambda) F(\overline{\Omega} \cup H) , \quad \text{as } \{H\} = \mathcal{D}^2, \text{ where } H \text{ is unlinked from } \overline{\Omega} \\ &= \frac{1}{\mathcal{D}^2} \sum_{\lambda \in \text{col}(H)} \dim(\lambda) F(\overline{\Omega}') , \quad \begin{array}{l} \text{using the invariance of } \{L, \Omega\} \\ \text{under sliding see figure 3.2} \end{array} \\ &= \sum_{i \in I} \dim(V_i) \mathcal{D}^{-2} \sum_{j \in I} \dim(V_j) F(\overline{\Omega}') \\ &= \sum_{i \in I} \dim(V_i) \tau(S^2 \times S^1, \overline{1_{V_i} \otimes \Omega}) , \quad \text{by formula (3.1)} \\ &= \sum_{i \in I} \dim(V_i) \text{Trace}_{V(S^2, l+1)}(Z_{(S^2 \times I, 1_{V_i} \otimes \Omega)}) , \quad \text{by proposition (3.4.1)} \end{aligned} \tag{3.2}$$

■

Definition 3.4.1 Let $J_p = (p, \dim(V_i)^p - \dim(V_i))$ be the ideal generated by p and $\dim(V_i)^p - \dim(V_i), \forall i \in I$ in K .

Corollary 3.4.1 Let Ω be any colored ribbon graph over any modular category with integrally closed ground ring. Then

$$\text{Tr}(\Omega)^p \equiv \text{Tr}(\Omega^p) \pmod{J_p}. \tag{3.3}$$

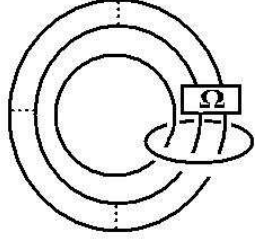


Figure 3.2: The graph $\overline{\Omega}'$ obtained by sliding $\overline{\Omega}$ over the Hopf link using the second Kirby move

Proof. If we assume

$$\dim(V_i)^p = \dim(V_i),$$

it follows that

$$\mathcal{D}^{2p} = \sum_{i \in I} \dim(V_i)^{2p} = \mathcal{D}^2 \pmod{p}.$$

Hence the result follows from Lemma (3.4.1) and [6, Lem. 3.5(i)], which implies that

$$\text{Trace}(Z^p) \equiv [\text{Trace}(Z)]^p \pmod{p},$$

where Z is an endomorphism of free K -module. ■

3.5 Quantum Invariants of Periodic 3-Manifolds

Let M be a 3-manifold that admits a \mathbb{Z}_p -action with a fixed point set equal to a circle. Then we are in situation of Theorem 3.3.1. i.e. M is obtained by surgery on S^3 along a framed p -periodic link L (see figure 3.1). We would like to relate the quantum invariant of M to the quantum invariant of $M_* = M/\mathbb{Z}_p$. Before we do so, we introduce the following.

Definition 3.5.1 *Let L be a p -periodic link, and λ be a coloring of L . If $\Gamma(L, \lambda)$ is invariant under the rotation of the graph that represents L by $2\pi/p$, then λ is called a p -periodic coloring.*

Lemma 3.5.1 *Let L be a p -periodic link, such that $L_* = L/\mathbb{Z}_p$. Then*

$$\{L\} \equiv \{L_*\}^p \pmod{J_p}. \quad (3.4)$$

Proof. Let us start with any coloring of L say λ , either λ is p -periodic or not. Let us assume that λ is not p -periodic, i.e $\Gamma(L, \lambda)$ is not invariant under the rotation by $2\pi/p$ about the z -axis. Hence the i -th rotation of $\Gamma(L, \lambda)$ (the rotation by $2i\pi/p$) represents a ribbon graph with the same value under F (since F is an isotopy invariant) and different coloring denoted by λ_i . So the term with a non-periodic coloring occurs p times. Hence we reduce the summation on the left-hand side to the periodic colorings. Now the result follows from corollary (3.4.1) and the fact that the periodic colorings of L are in one-to-one correspondence with the colorings of L_* (by restriction). ■

We introduce the notion $\kappa = \Delta\mathcal{D}^{-1}$. Now, we are ready to give a relation between the quantum invariants of M and M_* .

Theorem 3.5.1 *Over any modular category with integrally closed ground ring K ; we have*

$$I(M) \equiv \kappa^\delta I(M_*)^p \pmod{J_p}, \quad (3.5)$$

for some integer δ .

Proof. We assume that M and M_* are obtained by surgery on S^3 along L and L_* respectively.

$$\begin{aligned} I(M) &= (\Delta\mathcal{D}^{-1})^{\sigma(L)} \mathcal{D}^{-pm} \{L\} \\ &\equiv (\Delta\mathcal{D}^{-1})^{\sigma(L)} \mathcal{D}^{-pm} \{L_*\}^p \pmod{J_p} \text{ by lemma(3.5.1)} \\ &\equiv (\Delta\mathcal{D}^{-1})^{\sigma(L)-p\sigma(L_*)} ((\Delta\mathcal{D}^{-1})^{\sigma(L_*)})^p (\mathcal{D}^{-m})^p \{L_*\}^p \pmod{J_p} \\ &\equiv \kappa^\delta I(M_*)^p \pmod{J_p}. \end{aligned} \quad (3.6)$$

Here $\delta = \sigma(L) - p\sigma(L_*)$. ■

Corollary 3.5.1

$$\tau(M) \equiv \kappa^\delta \mathcal{D}^{p-1} \tau(M_*)^p \pmod{J_p}, \quad (3.7)$$

where δ and κ as defined before.

Before we go to the next corollary, we define the total signature for a knot.

Definition 3.5.2 Suppose K is a knot in a homology sphere M_* . Let $\pi : M \longrightarrow M_*$ be the p -fold cyclic cover branched along K . It is known that, we can extend this cover to a cover $W \longrightarrow W_*$ of 4-manifolds (where $M = \partial W$ and $M_* = \partial W_*$) branching over the surface Y . Let $Y \cdot Y$ denote the self-intersection of Y in W_* using the framing on K obtained from any Seifert surface for K in M_* . In this case, we define the total signature $\sigma_p(K)$

$$\begin{aligned}\sigma_p(K) &= p\sigma(W_*) - \sigma(W) - \frac{p^2 - 1}{3p} Y \cdot Y \\ &= p\sigma(L_*) - \sigma(L) - \frac{p^2 - 1}{3p} Y \cdot Y.\end{aligned}$$

By a well-known argument, using Novikov additivity and the G -signature theory [15], $\sigma_p(K)$ is independent of the choices made.

The following corollary generalizes [8, Th. 3].

Corollary 3.5.2 If M is a p -fold branched cyclic cover of a homology sphere along a knot K , then

$$I(M) \equiv \kappa^{-\sigma_p(K)} I(M_*)^p \pmod{J_p},$$

where $\sigma_p(K)$ is the total signature of K .

Proof. The linking matrix of L describes the intersection form on 4-manifold with boundary M which is a branched cover along a disk with zero self-intersection in a 4-manifold with boundary M_* . The corollary now follows by identifying δ with the total signature of K . ■

Corollary 3.5.3 If M is a p -fold branched cyclic cover of S^3 along the knot K , then

$$I(M) \equiv \kappa^{-\sigma_p(K)} \pmod{J_p}.$$

Remark 3.5.1 If K is a knot in S^3 , $\sigma_p(K)$ can be identified with the sum of the Tristram-Levine “ p -signatures”.

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Vita

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